

ON THE DERIVED DG FUNCTORS

VADIM VOLOGODSKY

ABSTRACT. Assume that abelian categories \mathcal{A} , \mathcal{B} over a field admit countable direct limits and that these limits are exact. Let $\mathcal{F} : D_{dg}^+(\mathcal{A}) \rightarrow D_{dg}^+(\mathcal{B})$ be a DG quasi-functor such that the functor $Ho(\mathcal{F}) : D^+(\mathcal{A}) \rightarrow D^+(\mathcal{B})$ carries $D^{\geq 0}(\mathcal{A})$ to $D^{\geq 0}(\mathcal{B})$ and such that, for every $i > 0$, the functor $H^i \mathcal{F} : \mathcal{A} \rightarrow \mathcal{B}$ is effaceable. We prove that \mathcal{F} is canonically isomorphic to the right derived DG functor $RH^0(\mathcal{F})$. We also prove a similar result for bounded derived DG categories in a more general setting. We give an example showing that the corresponding statements for triangulated functors are false. We prove a formula that expresses Hochschild cohomology of the categories $D_{dg}^b(\mathcal{A})$, $D_{dg}^+(\mathcal{A})$ as the *Ext* groups in the abelian category of left exact functors $\mathcal{A} \rightarrow Ind \mathcal{B}$.

1. MAIN RESULTS

Let \mathcal{A} and \mathcal{B} be abelian categories, and let

$$RF_{tri} : D^+(\mathcal{A}) \rightarrow D^+(\mathcal{B})$$

be the right derived functor of some left exact functor $F : \mathcal{A} \rightarrow \mathcal{B}$. Then, the corresponding cohomological δ -functor $R^*F = H^*RF_{tri} : \mathcal{A} \rightarrow \mathcal{B}$ has the following property: $H^i RF_{tri} = 0$ for $i < 0$, effaceable for $i > 0$ and $H^0 RF_{tri} \simeq F$. Conversely, according to a result of Grothendieck ([G]) every cohomological δ -functor $T^* : \mathcal{A} \rightarrow \mathcal{B}$ satisfying the above property is canonically isomorphic to the right derived functor R^*F . The purpose of this paper is to extend this extremely useful characterization of R^*F to the derived category level. Unfortunately, Verdier's notion of triangulated functor is too poor to allow such a simple characterization of the derived functors (see Remark 1.1). In order to get a meaningful statement one has to consider triangulated functors with some kind of enrichment. Arguably the most useful notion here is the one of *DG quasi-functor* (or essentially equivalent notion of A_∞ -functor). Indeed, works of Keller and Drinfeld ([K2], [Dri]) provide a canonical DG enhancement $D_{dg}^+(\mathcal{A})$ of Verdier's triangulated derived category. Roughly, a DG quasi-functor $\mathcal{F} : D_{dg}^b(\mathcal{A}) \rightarrow D_{dg}^b(\mathcal{B})$ is a diagram of the form

$$(1.1) \quad D_{dg}^+(\mathcal{A}) \xleftarrow{S} \mathcal{C} \xrightarrow{G} D_{dg}^+(\mathcal{B}),$$

where \mathcal{C} is a DG category, S , G DG functors, and S is a homotopy equivalence. Every quasi-functor (1.1) yields a triangulated functor $Ho(\mathcal{F}) : D^+(\mathcal{A}) \rightarrow D^+(\mathcal{B})$, but the converse is not true in general. Nevertheless, many of the natural triangulated functors come together with a DG enhancement. For example, the triangulated derived functor RF can be canonically promoted to a DG quasi-functor ([Dri] §5). The main result of this paper states that under certain mild assumptions on abelian categories \mathcal{A} and \mathcal{B} the DG quasi-functors isomorphic to the DG derived ones are precisely the DG quasi-functors satisfying Grothendieck's condition above. To state the result we need to introduce some notations.

Let k be a commutative ring. Denote by $Mod(k)$ the category of k -modules. We shall say that k -linear category ¹ is k -flat if, for every two objects X, Y , the k -module $Hom(X, Y)$ is flat. Given a k -linear exact category \mathcal{A} we denote by $D_{dg}^b(\mathcal{A})$ the corresponding bounded derived DG category over k . This the DG quotient ([Dri]) of the DG category $C_{dg}^b(\mathcal{A})$ of bounded complexes by the subcategory of acyclic ones ([N], §1). The homotopy category of $D_{dg}^b(\mathcal{A})$ is the triangulated derived category $D^b(\mathcal{A})$ as defined in ([N]). Let \mathcal{B} be another k -linear abelian category, $D_{dg}^b(\mathcal{B})$ the corresponding bounded derived DG category, and let $\mathcal{T}(D_{dg}^b(\mathcal{A}), D_{dg}^b(\mathcal{B}))$ be the triangulated category of DG quasi-functors $\mathcal{F} : D_{dg}^b(\mathcal{A}) \rightarrow D_{dg}^b(\mathcal{B})$ ([Dri], §16.1). Given such \mathcal{F} and an integer i we denote by $H^i \mathcal{F} : \mathcal{A} \rightarrow \mathcal{B}$ the composition

$$\mathcal{A} \rightarrow D_{dg}^b(\mathcal{A}) \xrightarrow{\mathcal{F}} D_{dg}^b(\mathcal{B}) \xrightarrow{H^i} \mathcal{B}.$$

Theorem 1. *Let \mathcal{A} be a small k -flat exact category and \mathcal{B} a small abelian k -linear category.*

(1) *Assume that a DG quasi-functor*

$$\mathcal{F} : D_{dg}^b(\mathcal{A}) \rightarrow D_{dg}^b(\mathcal{B})$$

has the following property:

(P) *The functors $H^i \mathcal{F} : \mathcal{A} \rightarrow \mathcal{B}$ are 0 for every $i < 0$ and effaceable (i.e., for every object $X \in \mathcal{A}$, there is an admissible monomorphism $X \hookrightarrow Y$ such that the induced morphism $H^i \mathcal{F}(X) \rightarrow H^i \mathcal{F}(Y)$ is 0) for every $i > 0$.*

Then the functor $F := H^0 \mathcal{F} : \mathcal{A} \rightarrow \mathcal{B}$ is left exact, has a right derived DG quasi-functor ([Dri] §5)

$$RF : D_{dg}^b(\mathcal{A}) \rightarrow D_{dg}^b(\mathcal{B}),$$

and there is a unique isomorphism $\mathcal{F} \simeq RF$ such that the induced automorphism $F = H^0(\mathcal{F}) \simeq H^0(RF) = F$ equals Id. Conversely, the right derived DG quasi-functor of any left exact functor $F : \mathcal{A} \rightarrow \mathcal{B}$ satisfies property (P).

(2) *For every two DG quasi-functors $\mathcal{F}, \mathcal{G} \in \mathcal{T}(D_{dg}^b(\mathcal{A}), D_{dg}^b(\mathcal{B}))$ satisfying property (P) and every $i < 0$*

$$Hom_{\mathcal{T}(D_{dg}^b(\mathcal{A}), D_{dg}^b(\mathcal{B}))}(\mathcal{F}, \mathcal{G}[i]) = 0,$$

$$Hom_{\mathcal{T}(D_{dg}^b(\mathcal{A}), D_{dg}^b(\mathcal{B}))}(\mathcal{F}, \mathcal{G}) = Hom_{Fct(\mathcal{A}, \mathcal{B})}(H^0 \mathcal{F}, H^0 \mathcal{G}).$$

Here $Fct(\mathcal{A}, \mathcal{B})$ denotes the category of all k -linear functors $\mathcal{A} \rightarrow \mathcal{B}$.

Remark 1.1. The analogous statement for triangulated functors is false (in contrast to DG enhanced functors). Here is an example of a triangulated functor $\mathcal{F}_{tri} : D^b(Mod(\mathbb{C}[x])) \rightarrow D^b(Mod(\mathbb{C}[x]))$ such that $H^i \mathcal{F}_{tri}$ is 0 for $i < 0$ and effaceable for $i > 0$ but \mathcal{F}_{tri} is not isomorphic (as a triangulated functor) to a right derived functor. Recall ([KS], §10.1.9) that a triangulated functor \mathcal{F}_{tri} is a pair $(\mathcal{F}_{add}, \tau)$, where \mathcal{F}_{add} is an additive functor and τ is an isomorphism of functors $\mathcal{F}_{add} \circ T \simeq T \circ \mathcal{F}_{add}$ (here T is the translation functor: $T(X) = X[1]$), preserving the class of distinguished triangles. Consider $\mathcal{F}_{tri} = (Id, \tau) : D^b(Mod(\mathbb{C}[x])) \rightarrow D^b(Mod(\mathbb{C}[x]))$, where $\tau : Id \circ T = T \xrightarrow{-Id} T = T \circ Id$ is the multiplication by -1 . We claim that \mathcal{F}_{tri} is

¹i.e., a category enriched over $Mod(k)$.

not isomorphic to (Id, Id) . Indeed, such isomorphism would be given by an automorphism S of the identity functor $Id : D^b(Mod(\mathbb{C}[x])) \rightarrow D^b(Mod(\mathbb{C}[x]))$ with the following property: for every $M \in D^b(Mod(\mathbb{C}[x]))$

$$S(M)[1] = -S(M[1]) : M[1] \rightarrow M[1].$$

Taking $M = \mathbb{C}$ and observing that $Hom(\mathbb{C}, \mathbb{C}[1]) = \mathbb{C}$ we see that no such S may exist.

Remark 1.2. It is likely that the k -flatness assumption on \mathcal{A} is abundant. However, I can not prove this.

We have a similar result for bounded from below derived DG categories. If \mathcal{A} is a k -linear abelian category we will write $D_{dg}^+(\mathcal{A})$ for the bounded from below derived DG category of \mathcal{A} and $D^+(\mathcal{A})$ for the corresponding triangulated category. Let $D^{\geq n}(\mathcal{A})$ be the full subcategory of $D^+(\mathcal{A})$ that consists of complexes with trivial cohomology in degrees less than n . We say that a DG quasi-functor

$$\mathcal{F} : D_{dg}^+(\mathcal{A}) \rightarrow D_{dg}^+(\mathcal{B})$$

has property (P') if

(P') $Ho(\mathcal{F})(D^{\geq 0}(\mathcal{A})) \subset D^{\geq 0}(\mathcal{B})$ and, for every $i > 0$, the functor $H^i \mathcal{F} : \mathcal{A} \rightarrow \mathcal{B}$ is effaceable.

Theorem 2. *Let k be a field and let \mathcal{A}, \mathcal{B} be small abelian k -linear categories. Assume that the both categories are closed under countable direct limits and that these limits are exact.*

- (1) *Let $\mathcal{F} \in \mathcal{T}(D_{dg}^+(\mathcal{A}), D_{dg}^+(\mathcal{B}))$ be a DG quasi-functor satisfying property (P') and $F := H^0 \mathcal{F} : \mathcal{A} \rightarrow \mathcal{B}$. The functor F admits a right derived DG quasi-functor $R\mathcal{F} : D_{dg}^+(\mathcal{A}) \rightarrow D_{dg}^+(\mathcal{B})$ and there is a unique isomorphism $\mathcal{F} \simeq R\mathcal{F}$ such that the induced automorphism $F = H^0(\mathcal{F}) \simeq H^0(R\mathcal{F}) = F$ equals Id . Conversely, a right derived DG quasi-functor of any left exact functor $F : \mathcal{A} \rightarrow \mathcal{B}$ satisfies property (P') .*
- (2) *For every two DG quasi-functors $\mathcal{F}, \mathcal{G} \in \mathcal{T}(D_{dg}^+(\mathcal{A}), D_{dg}^+(\mathcal{B}))$ satisfying property (P') and every $i < 0$*

$$Hom_{\mathcal{T}(D_{dg}^+(\mathcal{A}), D_{dg}^+(\mathcal{B}))}(\mathcal{F}, \mathcal{G}[i]) = 0,$$

$$Hom_{\mathcal{T}(D_{dg}^+(\mathcal{A}), D_{dg}^+(\mathcal{B}))}(\mathcal{F}, \mathcal{G}) = Hom_{Fct(\mathcal{A}, \mathcal{B})}(H^0 \mathcal{F}, H^0 \mathcal{G}).$$

The main ingredient of the proof of Theorem 2 is the following construction. Let $Sh(\mathcal{A}^o \otimes_k \mathcal{B})$ be the category of k -linear contravariant functors $\mathcal{A}^o \otimes_k \mathcal{B} \rightarrow Mod(k)$ that are left exact with respect to the both arguments. Every k -linear left exact functor $F : \mathcal{A} \rightarrow \mathcal{B}$ yields $s(F) \in Sh(\mathcal{A}^o \otimes_k \mathcal{B})$:

$$s(F)(X \otimes X') = Hom_{\mathcal{B}}(X', F(X)).$$

Let $\mathcal{T}^+ \subset \mathcal{T}(D_{dg}^+(\mathcal{A}), D_{dg}^+(\mathcal{B}))$ be the full triangulated subcategory whose objects are quasi-functors \mathcal{F} such that $Ho(\mathcal{F})(D^{\geq 0}(\mathcal{A})) \subset D^{\geq n}(\mathcal{B})$ for some n . Using key Lemma 2.1 we construct a fully faithful embedding

$$(1.2) \quad \mathcal{T}^+ \hookrightarrow D(Sh(\mathcal{A}^o \otimes_k \mathcal{B}))$$

that carries every DG quasi-functor \mathcal{F} satisfying property (P') to $s(F) \in Sh(\mathcal{A}^o \otimes_k \mathcal{B}) \subset D(Sh(\mathcal{A}^o \otimes_k \mathcal{B}))$.

As another application of (1.2) we compute the Hochschild cohomology of the derived DG category. Recall (see, e.g. [K1], §5.4) that the Hochschild cohomology of a DG category \mathcal{C} can be interpreted as

$$(1.3) \quad HH^i(\mathcal{C}, \mathcal{C}) = \text{Hom}_{\mathcal{T}(\mathcal{C}, \mathcal{C})}(Id_{\mathcal{C}}, Id_{\mathcal{C}}[i]).$$

The composition in \mathcal{C} makes $HH^*(\mathcal{C}, \mathcal{C})$ a graded commutative algebra over k .

Theorem 3. *Let k be a field and let \mathcal{A} be a small abelian k -linear category. There is an isomorphism of algebras*

$$(1.4) \quad HH^*(D_{dg}^b(\mathcal{A}), D_{dg}^b(\mathcal{A})) \simeq \text{Ext}_{Sh(\mathcal{A}^o \otimes_k \mathcal{A})}^*(s(Id_{\mathcal{A}}), s(Id_{\mathcal{A}})).$$

If, in addition, \mathcal{A} is closed under countable direct limits and that these limits are exact

$$(1.5) \quad HH^*(D_{dg}^+(\mathcal{A}), D_{dg}^+(\mathcal{A})) \simeq \text{Ext}_{Sh(\mathcal{A}^o \otimes_k \mathcal{A})}^*(s(Id_{\mathcal{A}}), s(Id_{\mathcal{A}})).$$

Remark 1.3. The category $Sh(\mathcal{A}^o \otimes_k \mathcal{A})$ has a tensor structure that extends the tensor structure on the category of left exact endofunctors $\mathcal{A} \rightarrow \mathcal{A}$ given by the composition. This can be used to promote (1.4), (1.5) to isomorphisms of *Gerstenhaber algebras* (see, e.g. [K1], §5.4).

Notation. Given a category \mathcal{C} we denote by \mathcal{C}^o the opposite category. If \mathcal{C} is a DG category we will write $Ho\mathcal{C}$ for the corresponding homotopy category ([Dri], §2.7). For example, $HoC(Mod(k))$ denotes the homotopy category of complexes of k -modules. The derived category of right DG modules over a DG category \mathcal{C} will be denoted by $\mathbb{D}(\mathcal{C})$ ([Dri], §2.3)². We will write $\underline{\mathcal{C}}$ for the DG category of semi-free DG \mathcal{C}^o -modules ([BV], 1.6.1). We have a canonical equivalence of triangulated categories $Ho\underline{\mathcal{C}} \xrightarrow{\sim} \mathbb{D}(\mathcal{C})$ ([BV], 1.6.4). For DG categories $\mathcal{C}, \mathcal{C}'$ we denote by $\mathcal{T}(\mathcal{C}, \mathcal{C}')$ the category of DG quasi-functors ([Dri], §16.1). If \mathcal{C}' is a pretriangulated ([Dri], §2.4) $\mathcal{T}(\mathcal{C}, \mathcal{C}')$ has a canonical structure of triangulated category. If $\mathcal{F} \in \mathcal{T}(\mathcal{C}, \mathcal{C}')$ we will write $Ho(\mathcal{F})$ for the corresponding functor between the homotopy categories.

Acknowledgements. I would like to thank Sasha Beilinson, Bernhard Keller, and Dima Orlov for helpful conversations related to the subject of this paper. This research was partially supported by NSF grant DMS-0901707.

2. PROOFS

Proof of theorem 1. Let $\mathcal{T}^+ \subset \mathcal{T} := \mathcal{T}(D_{dg}^b(\mathcal{A}), D_{dg}^b(\mathcal{B}))$ be the full triangulated subcategory whose objects are quasi-functors \mathcal{F} such that $H^i \mathcal{F} = 0$ for sufficiently small i . To prove Theorem we shall construct (in Lemma 2.1 below) a fully faithful embedding of \mathcal{T}^+ into derived category of a certain abelian category $Sh(\mathcal{A}^o \otimes_k \mathcal{B})$ that takes every functor $\mathcal{F} \in \mathcal{T}^+$ satisfying property (P) to an object of the heart $Sh(\mathcal{A}^o \otimes_k \mathcal{B}) \subset Sh(\mathcal{A}^o \otimes_k \mathcal{B})$.

Under our flatness assumption on \mathcal{A} , the category \mathcal{T} is a full subcategory of the derived category $\mathbb{D}(D_{dg}^b(\mathcal{A})^o \otimes_k D_{dg}^b(\mathcal{B}))$ of right DG modules over $D_{dg}^b(\mathcal{A})^o \otimes_k D_{dg}^b(\mathcal{B})$ that consists of all $M \in \mathbb{D}(D_{dg}^b(\mathcal{A})^o \otimes_k D_{dg}^b(\mathcal{B}))$ such that, for every X in $D_{dg}^b(\mathcal{A})^o$, the module $M(X) \in \mathbb{D}(D_{dg}^b(\mathcal{B}))$ belongs to the essential image of the Yoneda embedding $D_{dg}^+(\mathcal{B}) \rightarrow \mathbb{D}(D_{dg}^b(\mathcal{B}))$ ([Dri], §16.1).

²Drinfeld's notation for this category is $D(\mathcal{C})$. We use $\mathbb{D}(\mathcal{C})$ to avoid confusion with Verdier's derived category of an abelian category \mathcal{C} that is denoted by $D(\mathcal{C})$.

Consider the restriction functor

$$\mathbb{D}(D_{dg}^b(\mathcal{A})^o \otimes_k D_{dg}^b(\mathcal{B})) \xrightarrow{\beta} \mathbb{D}(\mathcal{A}^o \otimes_k \mathcal{B})$$

induced by the DG quasi-functor $\mathcal{A}^o \otimes_k \mathcal{B} \rightarrow D_{dg}^b(\mathcal{A})^o \otimes_k D_{dg}^b(\mathcal{B})$. By definition, the triangulated category $\mathbb{D}(\mathcal{A}^o \otimes_k \mathcal{B})$ is the derived category of the abelian category $PSh := PSh(\mathcal{A}^o \otimes_k \mathcal{B})$ of k -linear presheaves *i.e.*, the category of k -linear contravariant functors $\mathcal{A}^o \otimes_k \mathcal{B} \rightarrow Mod(k)$. Consider a Grothendieck topology on $\mathcal{A}^o \otimes_k \mathcal{B}$ whose covers are maps of the form $f \otimes g : Y \otimes Y' \rightarrow X \otimes X'$, where $X, Y \in \mathcal{A}^o$, $X', Y' \in \mathcal{B}$, and $f : Y \rightarrow X$, $g : Y' \rightarrow X'$ are admissible epimorphisms³ *i.e.*, a sieve \mathcal{C} over $X \otimes X'$ is a covering sieve if there exist $f : Y \rightarrow X$, $g : Y' \rightarrow X'$ as above such that $Y \otimes Y' \xrightarrow{f \otimes g} X \otimes X' \in \mathcal{C}$. The axioms of Grothendieck topology (see, e.g. [KS], §16.1) are immediate except for the one which is the following statement: for every cover $Y \otimes Y' \xrightarrow{f \otimes g} X \otimes X'$ and every morphism $Z \otimes Z' \xrightarrow{\phi} X \otimes X'$ there exists a cover $T \otimes T' \xrightarrow{p \otimes q} Z \otimes Z'$ and a morphism such that $T \otimes T' \xrightarrow{\psi} Y \otimes Y'$ such that $(f \otimes g) \circ \psi = \phi \circ (p \otimes q)$, which is a consequence of the base change axiom of exact category ([Q], §2). Let $Sh := Sh(\mathcal{A}^o \otimes_k \mathcal{B})$ be the subcategory of PSh that consists of objects satisfying the sheaf property. Explicitly, objects of the category $Sh(\mathcal{A}^o \otimes_k \mathcal{B})$ are contravariant functors $\mathcal{A}^o \otimes_k \mathcal{B} \rightarrow Mod(k)$ that are left exact with respect to the both arguments. The embedding $Sh \rightarrow PSh$ has a left adjoint functor (sheafification)

$$\sim : PSh \rightarrow Sh,$$

which is exact ([KS], §17.4). We denote by $\gamma : D(PSh) \rightarrow D(Sh)$ the induced functor between the derived categories. The composition

$$\mathbb{D}(D_{dg}^b(\mathcal{A})^o \otimes_k D_{dg}^b(\mathcal{B})) \xrightarrow{\beta} D(PSh) \xrightarrow{\gamma} D(Sh)$$

is *not* fully faithful in general, however, we have the following result.

Lemma 2.1. *Let $\mathbb{D}^+ \subset \mathbb{D}(D_{dg}^b(\mathcal{A})^o \otimes_k D_{dg}^b(\mathcal{B}))$ be the full subcategory whose objects are DG modules M such that $\beta(M) \in D^+(PSh)$. Then the functor*

$$S : \mathbb{D}^+ \xrightarrow{\beta} D^+(PSh) \xrightarrow{\gamma} D^+(Sh)$$

is an equivalence of categories.

Proof. The category $D_{dg}^b(\mathcal{A})^o \otimes_k D_{dg}^b(\mathcal{B})$ is a DG quotient of the category $C_{dg}^b(\mathcal{A})^o \otimes_k C_{dg}^b(\mathcal{B})$ by the full subcategory whose objects are of the form $X^\cdot \otimes X'^\cdot$, where either X^\cdot or X'^\cdot is acyclic. It then follows from ([Dri], Theorem 1.6.2) that the functor

$$\beta : \mathbb{D}(D_{dg}^b(\mathcal{A})^o \otimes_k D_{dg}^b(\mathcal{B})) \rightarrow \mathbb{D}(C_{dg}^b(\mathcal{A})^o \otimes_k C_{dg}^b(\mathcal{B})) = D(PSh)$$

is fully faithful and that its essential image consists of all DG-modules $M \in \mathbb{D}(C_{dg}^b(\mathcal{A})^o \otimes_k C_{dg}^b(\mathcal{B}))$ that carry every $X^\cdot \otimes X'^\cdot$ with the above property to an acyclic complex. Identifying the category $\mathbb{D}(C_{dg}^b(\mathcal{A})^o \otimes_k C_{dg}^b(\mathcal{B}))$ with $D(PSh)$ and observing that the subcategories of acyclic complexes in the homotopy categories $HoC_{dg}^b(\mathcal{A})$, $HoC_{dg}^b(\mathcal{B})$ are generated by short exact sequences ([N], §1) we exhibit $\mathbb{D}(D_{dg}^b(\mathcal{A})^o \otimes_k D_{dg}^b(\mathcal{B}))$ as a full subcategory $\mathcal{R} \subset D(PSh)$ whose objects are complexes F^\cdot of presheaves satisfying the following two conditions:

³By definition, admissible epimorphisms $Y \rightarrow X$ in \mathcal{A}^o are admissible monomorphisms $X \rightarrow Y$ in \mathcal{A} .

- For any exact sequence $0 \rightarrow Z \rightarrow Y \rightarrow X \rightarrow 0$ in \mathcal{A}^o and any $X' \in \mathcal{B}$ the total complex of

$$(2.1) \quad F^\bullet(X \otimes X') \rightarrow F^\bullet(Y \otimes X') \rightarrow F^\bullet(Z \otimes X')$$

is acyclic.

- For any $X \in \mathcal{A}^o$ and any exact sequence $0 \rightarrow Z' \rightarrow Y' \rightarrow X' \rightarrow 0$ in \mathcal{B} the total complex of

$$F^\bullet(X \otimes X') \rightarrow F^\bullet(X \otimes Y') \rightarrow F^\bullet(X \otimes Z')$$

is acyclic.

Observe that, for every $F^\bullet \in \mathcal{R}$ and an exact sequence $0 \rightarrow Z \rightarrow Y \rightarrow X \rightarrow 0$ in \mathcal{A}^o , we have a long exact sequence of k -modules

$$(2.2) \quad \cdots H^{m-1}(F^\bullet(Z \otimes X')) \rightarrow H^m(F^\bullet(X \otimes X')) \rightarrow H^m(F^\bullet(Y \otimes X')) \rightarrow H^m(F^\bullet(Z \otimes X')) \rightarrow \cdots$$

The equivalence of categories

$$\beta : \mathbb{D}(D_{dg}^b(\mathcal{A})^o \otimes_k D_{dg}^b(\mathcal{B})) \xrightarrow{\sim} \mathcal{R} \subset D(PSh)$$

carries \mathbb{D}^+ to the subcategory \mathcal{R}^+ of \mathcal{R} that consists of bounded from below complexes.

The derived category of sheaves $D(Sh)$ is the quotient of the derived category of presheaves by the subcategory $\mathcal{I}_{lac} \subset D(PSh)$ of locally (for our Grothendieck topology on $\mathcal{A}^o \otimes_k \mathcal{B}$) acyclic complexes ([BV], §1.11). We shall prove that

$$(2.3) \quad \mathcal{R}^+ \subset \mathcal{I}_{lac}^\perp,$$

where \mathcal{I}_{lac}^\perp denotes the right orthogonal complement to \mathcal{I}_{lac} in $D(PSh)$ ([BV] §1.1); *i.e.*

$$(2.4) \quad Hom_{D(PSh)}(G^\bullet, F^\bullet) = 0.$$

for every $G^\bullet \in \mathcal{I}_{lac}$ and $F^\bullet \in \mathcal{R}^+$. Without loss of generality we may assume that F^\bullet has trivial cohomology in negative degrees: $F^\bullet = F^0 \rightarrow F^1 \rightarrow \cdots$. Let $\tilde{F}^\bullet = \tilde{F}^0 \rightarrow \tilde{F}^1 \rightarrow \cdots$ be the corresponding complex of sheaves. Since the category of sheaves has enough injective objects (see, e.g. [KS], Theorems 9.6.2, 18.1.6) there exists a complex $I^\bullet = I^0 \rightarrow I^1 \rightarrow \cdots$ of injective sheaves together with a morphism $\tilde{F}^\bullet \rightarrow I^\bullet$ which is an isomorphism in the derived category of sheaves. Let us show that the composition

$$\delta : F^\bullet \rightarrow \tilde{F}^\bullet \rightarrow I^\bullet$$

is an isomorphism in the category derived category of presheaves. Indeed, every injective sheaf, viewed as a presheaf, is an object of \mathcal{R} . Thus I^\bullet and $cone(\delta)$ are in \mathcal{R}^+ . Assuming that $cone(\delta) \neq 0$ choose the smallest integer m such that

$$0 \neq H^m(cone(\delta)) \in PSh.$$

Then there exist an object $X \otimes X' \in \mathcal{A}^o \otimes_k \mathcal{B}$ and a nonzero element $a \in H^m(cone(\delta))(X \otimes X')$. Since the sheafification of $H^m(cone(\delta))$ is 0 there exists a cover $p : Y \otimes Y' \rightarrow X \otimes X'$ such that

$$0 = p^*a \in H^m(cone(\delta))(Y \otimes Y').$$

Writing p as a composition

$$Y \otimes Y' \xrightarrow{1 \otimes q} Y \otimes X' \xrightarrow{f \otimes 1} X \otimes X'$$

we may assume $(f \otimes 1)^*a = 0$ (otherwise, we replace $X \otimes X'$ by $Y \otimes X'$). Let us look at a fragment of the long exact sequence (2.2) applied to $F = \text{cone}(\delta)$ and the exact sequence $0 \rightarrow Z \rightarrow Y \xrightarrow{f} X \rightarrow 0$:

$$H^{m-1}(\text{cone}(\delta))(Z \otimes X') \rightarrow H^m(\text{cone}(\delta))(X \otimes X') \rightarrow H^m(\text{cone}(\delta))(Y \otimes X').$$

Since by our assumption $H^{m-1}(\text{cone}(\delta)) = 0$, it follows that $(f \otimes 1)^*$ is injective and, hence, $a = 0$. This contradiction proves that $\text{cone}(\delta) = 0$ i.e., δ is a quasi-isomorphism. Thus, to complete the proof of (2.4) it suffices to show that

$$\text{Hom}_{D(PSh)}(G, I) = 0,$$

for every $G \in \mathcal{I}_{lac}$ and every bounded from below complex of injective sheaves I . Indeed, every morphism $h : G \rightarrow I$ in the derived category is represented by a diagram in $C(PSh(\mathcal{A}^\circ \otimes_k \mathcal{B}))$

$$G \leftarrow G' \xrightarrow{h'} I,$$

where the first arrow is a quasi-isomorphism (and, in particular, $G' \in \mathcal{I}_{lac}$). If h' is homotopic to 0 then $h = 0$ in the derived category. Thus, it is enough to show that

$$\text{Hom}_{K(PSh)}(G', I) = 0$$

where $K(PSh)$ denotes the homotopy category of complexes. We have

$$\text{Hom}_{K(PSh)}(G', I) \xrightarrow{\sim} \text{Hom}_{K(Sh)}(\tilde{G}', I) \xrightarrow{\sim} \text{Hom}_{D(Sh)}(\tilde{G}', I).$$

The first arrow is an isomorphism because all terms of the complex I are sheaves; the second arrow is an isomorphism by ([KS], Lemma 13.2.4). Finally, $\text{Hom}_{D(Sh)}(\tilde{G}', I) = 0$ because the sheafification \tilde{G}' is 0 in $D(Sh)$.

To finish the proof of the lemma, we observe that, for every triangulated category \mathcal{C} and its full triangulated subcategory \mathcal{I} , the composition

$$\mathcal{I}^\perp \rightarrow \mathcal{C} \rightarrow \mathcal{C}/\mathcal{I}$$

is a fully faithful embedding: for every $X, Y \in \mathcal{C}$

$$\text{Hom}_{\mathcal{C}/\mathcal{I}}(X, Y) := \text{colim}_{f: X' \rightarrow X} \text{Hom}_{\mathcal{C}}(X', Y),$$

where the colimit is taken over the filtrant category of pairs $(X' \in \mathcal{C}, f : X' \rightarrow X)$ such that $\text{cone } f \in \mathcal{I}$. If $Y \in \mathcal{I}^\perp$, then

$$\text{Hom}_{\mathcal{C}}(X, Y) \xrightarrow{\sim} \text{Hom}_{\mathcal{C}}(X', Y),$$

and, hence,

$$\text{Hom}_{\mathcal{C}/\mathcal{I}}(X, Y) = \text{Hom}_{\mathcal{C}}(X, Y).$$

Applying this remark to $\mathcal{C} = D(PSh)$, $\mathcal{I} = \mathcal{I}_{lac}$ and using (2.4) we conclude that the functor $\mathcal{R}^+ \xrightarrow{\gamma} D(Sh)$ is fully faithful and, hence, so is the composition $\mathbb{D}^+ \xrightarrow{\sim} \mathcal{R}^+ \xrightarrow{\gamma} D(Sh)$. The essentially image the functor $\mathcal{R}^+ \xrightarrow{\gamma} D(Sh)$ is $D^+(Sh)$ because every complex of injective sheaves viewed as a complex of presheaves is on object of \mathcal{R}^+ . \square

Remark 2.2. Applying Lemma 2.1 to $k = \mathbb{Z}$ and $\mathcal{A} =$ the category of free abelian groups of finite rank we obtain the following statement: for every small abelian category \mathcal{B}

$$\mathbb{D}^+(D_{dg}^b(\mathcal{B})) \xrightarrow{\sim} D^+(PSh(\mathcal{B})) = D^+(Ind(\mathcal{B})),$$

where $\mathbb{D}^+(D_{dg}^b(\mathcal{B}))$ is a full subcategory of $\mathbb{D}(D_{dg}^b(\mathcal{B}))$ that maps to $D^+(PSh(\mathcal{B}))$ under the restriction functor (and the ind-completion $Ind(\mathcal{B})$ is just another name for $PSh(\mathcal{B})$ ([KS], §8.6)). Note the functor

$$(2.5) \quad \mathbb{D}(D_{dg}^b(\mathcal{B})) \rightarrow D(Ind(\mathcal{B}))$$

is not an equivalence of categories in general. In fact, the functor (2.5) factors as

$$(2.6) \quad \mathbb{D}(D_{dg}^b(\mathcal{B})) \xrightarrow{\phi} HoC(Ind(\mathcal{B}))/Ho\overline{C_{ac}^b(\mathcal{B})} \xrightarrow{p} D(Ind(\mathcal{B})),$$

where $Ho\overline{C_{ac}^b(\mathcal{B})}$ is the smallest triangulated subcategory of the homotopy category of acyclic complexes $HoC_{ac}(Ind(\mathcal{B}))$ that contains *finite* acyclic complexes $HoC_{ac}^b(\mathcal{B})$ and closed under arbitrary direct sums; the functor p is the projection

$$HoC(Ind(\mathcal{B}))/Ho\overline{C_{ac}^b(\mathcal{B})} \rightarrow HoC(Ind(\mathcal{B}))/HoC_{ac}(Ind(\mathcal{B})).$$

The equivalence ϕ can be constructed as follows. Let $\overline{C_{ac}^b(\mathcal{B})}$ be the full subcategory of the DG category $C(Ind(\mathcal{B}))$ whose objects are those of $Ho\overline{C_{ac}^b(\mathcal{B})}$. The DG quasi-functor $D_{dg}^b(\mathcal{B}) \rightarrow C(Ind(\mathcal{B}))/\overline{C_{ac}^b(\mathcal{B})}$ extends uniquely to a quasi-functor

$$\phi_{dg} : D_{dg}^b(\mathcal{B}) \rightarrow C(Ind(\mathcal{B}))/\overline{C_{ac}^b(\mathcal{B})}$$

that commutes with arbitrary direct sums ([BV], §1.6.1). Define

$$\phi := Ho\phi_{dg}.$$

Let us show that ϕ is an equivalence of categories. The subcategory $Ho\overline{C_{ac}^b(\mathcal{B})} \subset HoC(Ind(\mathcal{B}))$ is generated by compact objects (e.g., objects of $HoC_{ac}^b(\mathcal{B})$); it follows that the projection $HoC(Ind(\mathcal{B})) \rightarrow HoC(Ind(\mathcal{B}))/Ho\overline{C_{ac}^b(\mathcal{B})}$ carries compact objects of $HoC(Ind(\mathcal{B}))$ to compact objects of the quotient category ([BV], §1.4.2). In particular, in the commutative diagram

$$\begin{array}{ccc} D_{dg}^b(\mathcal{B}) & = & D_{dg}^b(\mathcal{B}) \\ \downarrow i & & \downarrow j \\ \mathbb{D}(D_{dg}^b(\mathcal{B})) & \xrightarrow{\phi} & HoC(Ind(\mathcal{B}))/Ho\overline{C_{ac}^b(\mathcal{B})} \end{array}$$

the image of j consists of compact objects. The same is true for the image of i ([BV], §1.7). The functors i, j are fully faithful and their images generate the categories $\mathbb{D}(D_{dg}^b(\mathcal{B}))$, $HoC(Ind(\mathcal{B}))/Ho\overline{C_{ac}^b(\mathcal{B})}$ respectfully. It follows that ϕ is an equivalence of categories.

In general, (e.g., if \mathcal{B} is the category of finitely generated modules over a finite group) the projection p is not conservative. However, if the category \mathcal{B} has *finite homological dimension* the objects of $D_{dg}^b(\mathcal{B})$ are compact in $D_{dg}^b(Ind(\mathcal{B}))$ ⁴ and the above argument proves that (2.5) is an equivalence of categories.

Corollary 2.3. *The composition*

$$(2.7) \quad S : \mathcal{T}^+ \xrightarrow{\alpha} \mathbb{D}(D_{dg}^b(\mathcal{A})^o \otimes_k D_{dg}^b(\mathcal{B})) \xrightarrow{\beta} D(PSh) \xrightarrow{\gamma} D(Sh)$$

is a fully faithful embedding.

⁴Indeed, under our finiteness assumption every complex in $D_{dg}^b(\mathcal{B})$ is quasi-isomorphic to a finite complex of projective objects. Thus it is enough to show that every projective object of \mathcal{B} is compact in $D(Ind(\mathcal{B}))$. This is clear because every such object is projective and compact in $Ind(\mathcal{B})$.

Consider the Yoneda embedding

$$s : \text{Fun}(\mathcal{A}, \mathcal{B}) \rightarrow \text{PSh}$$

that takes a functor $F \in \text{Fun}(\mathcal{A}, \mathcal{B})$ to the presheaf

$$s(F)(X \times X') = \text{Hom}_{\mathcal{B}}(X', F(X)).$$

If F is left exact then $s(F)$ is actually a sheaf.

Let $\mathcal{F} \in \mathcal{T}$ be a DG quasi-functor satisfying property (P). It follows from the definition of \mathcal{T}^+ given at the beginning of this section that $\mathcal{F} \in \mathcal{T}^+$. We shall prove that $S(\mathcal{F}) \xrightarrow{\sim} s(H^0\mathcal{F})$. Having in mind applications to Theorem 2 we will actually show a slightly more general statement. Namely, let us extend the functor (2.7) to a larger category:

$$S' : \mathcal{T}(D_{dg}^b(\mathcal{A}), D_{dg}^+(\mathcal{B})) \xrightarrow{\alpha'} \mathbb{D}(D_{dg}^b(\mathcal{A})^o \otimes_k D_{dg}^+(\mathcal{B})) \xrightarrow{\beta'} D(\text{PSh}) \xrightarrow{\gamma} D(\text{Sh}).$$

Lemma 2.4. *Let $\mathcal{F} \in \mathcal{T}(D_{dg}^b(\mathcal{A}), D_{dg}^+(\mathcal{B}))$ be a DG quasi-functor such that $H^i\mathcal{F}$ is zero for $i < 0$ and effaceable for $i > 0$. Set $s(F) = s(H^0\mathcal{F}) \subset \text{Sh} \subset D(\text{Sh})$ ⁵. Then $S'(\mathcal{F}) \in D(\text{Sh})$ is canonically isomorphic to $s(F)$.*

Proof. By definition, the cohomology presheaves of the complex $\beta'\alpha'(\mathcal{F}) \in D(\text{PSh})$ are given by the formula

$$H^i(\beta'\alpha'\mathcal{F})(X \otimes X') = \text{Hom}_{D^+(\mathcal{B})}(X', \text{Ho}(\mathcal{F})(X)[i]).$$

Since the negative cohomology of the complex $\text{Ho}(\mathcal{F})(X) \in D^+(\mathcal{B})$ vanish the same is true for $\beta'\alpha'\mathcal{F}$ and

$$H^0(\beta'\alpha'\mathcal{F})(X \otimes X') = \text{Hom}_{D^+(\mathcal{B})}(X', H^0\mathcal{F}(X)) = s(F).$$

It remains to prove that for $i > 0$ the sheafification of the presheaf $H^i(\beta'\alpha'\mathcal{F})$ equals zero. Given an integer j define presheaves $G^{i,j}$ to be

$$G^{i,j}(X \otimes X') = \text{Hom}_{D^+(\mathcal{B})}(X', \tau_{\leq j}(\text{Ho}(\mathcal{F})(X)))[i].$$

We shall show by induction on j that for every $i > 0$ and every j the sheafification of $G^{i,j}$ is 0. This would complete the proof since $G^{i,j} \xrightarrow{\sim} H^i(\beta'\alpha'\mathcal{F})(X \otimes X')$ for $j \geq i$. For every $i > 0$ and every element v of

$$G^{i,0}(X \otimes X') = \text{Ext}_{\mathcal{B}}^i(X', H^0\mathcal{F}(X))$$

there exists an epimorphism $Y' \rightarrow X'$ such that v is annihilated by the map

$$\text{Ext}_{\mathcal{B}}^i(X', H^0\mathcal{F}(X)) \rightarrow \text{Ext}_{\mathcal{B}}^i(Y', H^0\mathcal{F}(X))$$

([KS], Exercise 13.17). This proves that the sheafification of $G^{i,0}$ is 0. For the induction step, consider the distinguished triangle

$$\tau_{\leq j}(\text{Ho}(\mathcal{F})(X)) \rightarrow \tau_{\leq j+1}(\text{Ho}(\mathcal{F})(X)) \rightarrow H^{j+1}\mathcal{F}(X)[-j-1]$$

and the corresponding long exact sequence

$$\rightarrow G^{i,j}(X \otimes X') \rightarrow G^{i,j+1}(X \otimes X') \rightarrow \text{Hom}_{D^b(\mathcal{B})}(X', H^{j+1}\mathcal{F}(X)[-j-1+i]) \rightarrow .$$

It follows that $G^{i,j+1}$ fits in a long exact sequence

$$\rightarrow G^{i,j} \rightarrow G^{i,j+1} \rightarrow \text{Ext}_{\mathcal{B}}^{i-j-1}(\cdot, H^{j+1}\mathcal{F}(\cdot)) \rightarrow .$$

⁵The vanishing of $H^i\mathcal{F}$ implies that F is left exact and, hence, $s(F)$ is a sheaf.

The sheafification of $G^{i,j}$ is 0 by the induction assumption, the sheafification of $Ext_{\mathcal{B}}^{i-j-1}(\cdot, H^{j+1}\mathcal{F}(\cdot))$ is 0 because the functor $H^{j+1}\mathcal{F}$ is effaceable. Hence, the sheafification of $G^{i,j+1}$ is 0 as well. \square

Now we are ready to prove the second part of the theorem. Given quasi-functors $\mathcal{F}, \mathcal{G} \in \mathcal{T}$ satisfying property (P) we have by Lemmas 2.1, 2.4

$$(2.8) \quad Hom_{\mathcal{T}}(\mathcal{F}, \mathcal{G}[i]) \xrightarrow{\sim} Hom_{D(Sh)}(S(\mathcal{F}), S(\mathcal{G})[i]) \xrightarrow{\sim} Ext_{Sh}^i(s(H^0\mathcal{F}), s(H^0\mathcal{G})).$$

In particular, $Hom_{\mathcal{T}}(\mathcal{F}, \mathcal{G}[i])$ is isomorphic to $Hom_{Fun(\mathcal{A}, \mathcal{B})}(H^0\mathcal{F}, H^0\mathcal{G})$ if $i = 0$ (because $s : Fun(\mathcal{A}, \mathcal{B}) \rightarrow PSh$ is fully faithful) and to 0 for $i < 0$.

To prove the first part of the theorem we need to recall some facts about DG categories and derived functors. Let $f : \mathcal{C}_1 \rightarrow \mathcal{C}_2$ be a DG functor between small DG categories. Then the restriction functor $f_* : \mathbb{D}(\mathcal{C}_2) \rightarrow \mathbb{D}(\mathcal{C}_1)$ admits a left and a right adjoint functors (the derived induction and the co-induction functors)

$$(2.9) \quad f^*, f^! : \mathbb{D}(\mathcal{C}_1) \rightarrow \mathbb{D}(\mathcal{C}_2)$$

([Dri], §14.12). In particular, we have the canonical morphisms

$$(2.10) \quad \begin{aligned} Id &\rightarrow f_* f^*, & f_* f^! &\rightarrow Id \\ Id &\rightarrow f^! f_*, & f^* f_* &\rightarrow Id. \end{aligned}$$

It also follows from the adjunction property that f^* commutes with arbitrary direct sums and that $f^!$ commutes with arbitrary direct products. If the functor $Ho(f) : Ho(\mathcal{C}_1) \rightarrow Ho(\mathcal{C}_2)$ is fully faithful so is f_* and the first two morphisms in (2.10) are isomorphisms.

Recall the definition of the derived DG quasi-functor RF of a left exact functor $F : \mathcal{A} \rightarrow \mathcal{B}$ from ([Dri], §16). Consider the functor

$$\mathcal{T}(\mathcal{A}, D_{dg}^b(\mathcal{B})) \hookrightarrow \mathbb{D}(C_{dg}^b(\mathcal{A})^o \otimes_k D_{dg}^b(\mathcal{B})) \xrightarrow{f^*} \mathbb{D}(D_{dg}^b(\mathcal{A})^o \otimes_k D_{dg}^b(\mathcal{B}))$$

induced by the projection

$$f : C_{dg}^b(\mathcal{A})^o \otimes_k D_{dg}^b(\mathcal{B}) \rightarrow D_{dg}^b(\mathcal{A})^o \otimes_k D_{dg}^+(\mathcal{B}).$$

Given a k -linear functor $F \in Fun(\mathcal{A}, \mathcal{B}) \rightarrow \mathcal{T}(\mathcal{A}, D_{dg}^b(\mathcal{B}))$ define the “derived functor”

$$(2.11) \quad “RF” = f^*(F) \in \mathbb{D}(D_{dg}^b(\mathcal{A})^{op} \otimes_k D_{dg}^b(\mathcal{B})).$$

The right derived DG quasi-functor $RF : D_{dg}^b(\mathcal{A}) \rightarrow D_{dg}^b(\mathcal{B})$ if it exists is an object of $\mathcal{T}(D_{dg}^b(\mathcal{A}), D_{dg}^b(\mathcal{B}))$ whose image in $\mathbb{D}(D_{dg}^b(\mathcal{A})^o \otimes_k D_{dg}^b(\mathcal{B})) \supset \mathcal{T}(D_{dg}^b(\mathcal{A}), D_{dg}^b(\mathcal{B}))$ is “ RF ”.

Lemma 2.5. *Assume that F is left exact. Then “ RF ” $\in \mathbb{D}^+ \subset \mathbb{D}(D_{dg}^b(\mathcal{A})^{op} \otimes_k D_{dg}^b(\mathcal{B}))$ the functor $S : \mathbb{D}^+ \hookrightarrow D(Sh)$ takes “ RF ” to $s(F)$.*

Proof. Let $\beta : \mathbb{D}(D_{dg}^b(\mathcal{A})^o \otimes_k D_{dg}^b(\mathcal{B})) \rightarrow D(PSh)$ be the restriction functor, and let $\gamma : D(PSh) \rightarrow D(Sh)$ be the sheafification functor. As explained in ([Dri], §5) the presheaves $H^i(\beta(“RF”))$ can be computed as follows:

$$(2.12) \quad H^i(\beta(“RF”))(X \otimes X') = colim_Q Hom_{D^b(\mathcal{B})}(X', F(Y)[i]),$$

where the colimit is taken over the filtrant category Q of pairs $(Y, f) \in HoC_{dg}^b(\mathcal{A})$, $f \in Hom_{HoC_{dg}^b(\mathcal{A})}(X, Y)$ such that $cone(f)$ is acyclic. As the subcategory $Q' \subset Q$ consisting of pairs (Y, f) with $Y^j = 0$ for $j < 0$ is cofinal in Q the category Q in

equation (2.12) can be replaced by Q' . This proves that “ RF ” $\in \mathbb{D}^+$. Let us show that $\gamma \circ \beta(\text{“}RF\text{”}) \simeq s(F)$. We have

$$H^0(\beta(\text{“}RF\text{”}))(X \otimes X') = \text{colim}_{Q'} \text{Hom}_{D^b(\mathcal{B})}(X', F(Y')) \simeq$$

$$\text{colim}_{Q'} \text{Hom}_{D^b(\mathcal{B})}(X', \tau_{\leq 0} F(Y')) \simeq \text{colim}_{Q'} \text{Hom}_{D^b(\mathcal{B})}(X', F(X)) = s(F)(X \otimes X').$$

It remains to prove that, for every $i > 0$ the sheafification of $H^i(\beta(\text{“}RF\text{”}))$ is 0. Let s be a section of $H^i(\beta(\text{“}RF\text{”}))(X \otimes X')$ represented by an element

$$\tilde{s} \in \text{Hom}_{D^b(\mathcal{B})}(X', F(Y'))[i]$$

, where $X \xrightarrow{f} Y^0 \rightarrow Y^1 \rightarrow \dots$ is an object of Q' . Looking at the diagram

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y^0 & \rightarrow & Y^1 & \rightarrow & \dots \\ \downarrow f & & \downarrow Id & & \downarrow & & \\ Y^0 & \xrightarrow{Id} & Y^0 & \rightarrow & 0 & \rightarrow & \dots \end{array}$$

we see that the pullback $(f \otimes Id)^* s \in H^i(\beta(\text{“}RF\text{”}))(Y^0 \otimes X')$ is represented by an element of the group $\text{Hom}_{D^+(\mathcal{B})}(X', F(Y^0)[i]) = \text{Ext}_{\mathcal{B}}^i(X', F(Y^0))$. If $i > 0$ every element of this group is annihilated by the map $\text{Ext}_{\mathcal{B}}^i(X', F(Y^0)) \rightarrow \text{Ext}_{\mathcal{B}}^i(Y', F(Y^0))$ for some epimorphism $Y' \rightarrow X'$. \square

Let us prove the first part of the theorem. Let $\mathcal{F} \in \mathcal{T} \subset \mathbb{D}(D_{dg}^b(\mathcal{A})^o \otimes_k D_{dg}^b(\mathcal{B}))$ be a DF quasi-functor satisfying property (P) and $F = H^0 \mathcal{F}$. We need to construct an isomorphism $\mathcal{F} \simeq \text{“}RF\text{”}$. By Lemmas 2.4, 2.5 $\mathcal{F}, \text{“}RF\text{”} \in \mathbb{D}^+$. By Lemma 2.1 the functor $S : \mathbb{D}^+ \rightarrow D(Sh)$ is fully faithful. Thus, constructing an isomorphism $\mathcal{F} \simeq \text{“}RF\text{”}$ is equivalent to producing an isomorphism $S(\mathcal{F}) \simeq S(\text{“}RF\text{”})$ in $D(Sh)$ which is done in Lemmas 2.4, 2.5. Theorem 1 is proved.

Proof of theorem 2. Let $\mathcal{T}^+ \subset \mathcal{T} := \mathcal{T}(D_{dg}^+(\mathcal{A}), D_{dg}^+(\mathcal{B}))$ be the full triangulated subcategory whose objects are quasi-functors \mathcal{F} such that $\text{Ho}(\mathcal{F})(D^{\geq 0}(\mathcal{A})) \subset D^{\geq n}(\mathcal{B})$ for some n . We shall prove that the composition

$$\mathcal{T}^+ \hookrightarrow \mathbb{D}(D_{dg}^+(\mathcal{A})^o \otimes_k D_{dg}^+(\mathcal{B})) \xrightarrow{Res} \mathbb{D}(D_{dg}^b(\mathcal{A})^o \otimes_k D_{dg}^b(\mathcal{B})) \rightarrow D(Sh)$$

is a fully faithful embedding. Here Res denotes the restriction functor induced by the embedding

$$(2.13) \quad D_{dg}^b(\mathcal{A})^o \otimes_k D_{dg}^b(\mathcal{B}) \rightarrow D_{dg}^+(\mathcal{A})^o \otimes_k D_{dg}^+(\mathcal{B}).$$

To show this we need to introduce a bit of notation. If \mathcal{C} is an abelian category closed under countable direct sums and

$$X^0 \xrightarrow{\phi_0} X^1 \xrightarrow{\phi_1} X^2 \xrightarrow{\phi_2} \dots$$

is a diagram of complexes $X^i \in C(\mathcal{C})$ set

$$\text{hocolim } X^i = \text{cone}(\bigoplus_i X^i \xrightarrow{v} \bigoplus_i X^i) \in C(\mathcal{C}),$$

where $v|_{X^i} := Id_{X^i} - \phi_i : X^i \rightarrow \bigoplus_i X^i$. There is a canonical morphism

$$\text{hocolim } X^i \rightarrow \text{colim } X^i,$$

which is a quasi-isomorphism if countable direct limits in \mathcal{C} are exact. If this is the case, every morphism $X' \rightarrow X''$ of the diagrams that is a term-wise quasi-isomorphism

induces a quasi-isomorphism of the homotopy colimits ⁶. Dually, for a category \mathcal{C} closed under countable products and a diagram

$$\cdots \rightarrow X_2 \xrightarrow{\phi_1} X_1 \xrightarrow{\phi_0} X_0$$

set

$$\operatorname{holim} X_i = \operatorname{cone}\left(\prod_i X_i \xrightarrow{v} \prod_i X_i\right)[-1],$$

where $v_i := p_i - \phi_i p_{i+1} : \prod X_i \rightarrow X_i$ and $p_i : \prod X_i \rightarrow X_i$ are the projections.

Let $\mathbb{D}^f \subset \mathbb{D}(D_{dg}^+(\mathcal{A})^o \otimes_k D_{dg}^+(\mathcal{B}))$ be the full subcategory whose objects covariant DG functors $M : D_{dg}^+(\mathcal{A}) \otimes_k D_{dg}^+(\mathcal{B})^o \rightarrow C(\operatorname{Mod}(k))$ such that, for every $X \in D_{dg}^+(\mathcal{A})$ and $X' \in D_{dg}^+(\mathcal{B})$, the canonical morphism

$$(2.14) \quad M(X \otimes X') \rightarrow \operatorname{holim} M(X \otimes \tau_{<i} X'),$$

is a quasi-isomorphism and for every $X \in D_{dg}^+(\mathcal{A})$ and every *bounded* $X' \in D_{dg}^b(\mathcal{B})$ the canonical morphism

$$(2.15) \quad \operatorname{hocolim} M(\tau_{<i} X \otimes X') \rightarrow M(X \otimes X'),$$

is a quasi-isomorphism.

Remark 2.6. Since countable direct limits are exact in \mathcal{B} , the morphism $\operatorname{hocolim} \tau_{<i} X' \rightarrow X'$ is a quasi-isomorphism. Thus, property (2.14) is implied by the following: for every integer n and a countable collection $X^{n_i} \in D_{dg}^{\geq n}(\mathcal{B})$, the morphism

$$M(X \otimes \oplus_i X^{n_i}) \rightarrow \prod_i M(X \otimes X^{n_i})$$

is a quasi-isomorphism.

Remark 2.7. Since directed limits are exact in $\operatorname{Mod}(k)$ property (2.15) is equivalent to the following: for every $X \in D_{dg}^+(\mathcal{A})$ and $X' \in \mathcal{B}$,

$$(2.16) \quad \operatorname{colim} H^0(M(\tau_{<i} X \otimes X')) \xrightarrow{\sim} H^0(M(X \otimes X')).$$

Lemma 2.8. *Then the restriction functor*

$$\mathbb{D}^f \xrightarrow{\operatorname{Res}} \mathbb{D}(D_{dg}^b(\mathcal{A})^o \otimes_k D_{dg}^b(\mathcal{B}))$$

is an equivalence of categories.

Proof. We shall first consider the restriction

$$f_* : \mathbb{D}(D_{dg}^+(\mathcal{A})^o \otimes_k D_{dg}^+(\mathcal{B})) \rightarrow \mathbb{D}(D_{dg}^+(\mathcal{A})^o \otimes_k D_{dg}^b(\mathcal{B}))$$

and prove that $f^!$ and f_* define mutually inverse equivalences of categories

$$(2.17) \quad \mathbb{D}(D_{dg}^+(\mathcal{A})^o \otimes_k D_{dg}^b(\mathcal{B})) \simeq \mathbb{D}',$$

where \mathbb{D}' is a full subcategory of $\mathbb{D}(D_{dg}^+(\mathcal{A})^o \otimes_k D_{dg}^b(\mathcal{B}))$ whose objects are DG functors M satisfying property (2.14). Let us check that

$$(2.18) \quad f^!(\mathbb{D}(D_{dg}^+(\mathcal{A})^o \otimes_k D_{dg}^b(\mathcal{B}))) \subset \mathbb{D}'.$$

⁶For the last property, it suffices to assume that countable direct sums are exact in \mathcal{C} .

For every DG functor $f : \mathcal{C}_1 \rightarrow \mathcal{C}_2$ between DG categories over a field the functor $f^! : \mathbb{D}(\mathcal{C}_1) \rightarrow \mathbb{D}(\mathcal{C}_2)$ admits the following concrete description: if $M : \mathcal{C}_1 \rightarrow C(\text{Mod}(k))$ is a contravariant DG functor and $X \in \mathcal{C}_2$

$$(2.19) \quad f^!(M)(X) = \text{Hom}_{\mathbb{D}_{dg}(\mathcal{C}_1)}(f_*^{dg} \text{Hom}_{\mathcal{C}_2}(\cdot, X), M).$$

Here $\mathbb{D}_{dg}(\mathcal{C}_i)$ denotes the DG derived category of right \mathcal{C}_i -modules, f_*^{dg} the derived restriction functor, and $\text{Hom}_{\mathcal{C}_2}(\cdot, X)$ is the image of X under the Yoneda embedding $\mathcal{C}_2 \rightarrow \mathbb{D}_{dg}(\mathcal{C}_2)$.

We shall prove that

$$\text{hocolim} \text{Hom}(\cdot, X \otimes \tau_{<i} X') \rightarrow f_* \text{Hom}(\cdot, X \otimes X')$$

is an isomorphism in $\mathbb{D}(D_{dg}^+(\mathcal{A})^o \otimes_k D_{dg}^b(\mathcal{B}))$. Together with (2.19) it will imply (2.18). By definition of the tensor product of DG categories, for every $Y \otimes Y' \in D_{dg}^+(\mathcal{A})^o \otimes_k D_{dg}^b(\mathcal{B})$,

$$\text{Hom}(Y \otimes Y', X \otimes X') = \text{Hom}(Y, X) \otimes_k \text{Hom}(Y', X').$$

Hence, it is enough to check that

$$\text{hocolim} \text{Hom}_{D_{dg}^+(\mathcal{B})}(Y', \tau_{<i} Y) \rightarrow \text{Hom}_{D_{dg}^+(\mathcal{B})}(Y', Y)$$

is a quasi-isomorphism, for every $Y' \in D_{dg}^b(\mathcal{B})$. Using the exactness of direct limits in $\text{Mod}(k)$ the last assertion is reduced to the formula

$$\text{colim} \text{Hom}_{D^b(\mathcal{B})^o}(Y', \tau_{<i} Y) \simeq \text{Hom}_{D^+(\mathcal{B})^o}(Y', Y),$$

which holds because $\text{Hom}_{D^+(\mathcal{B})^o}(Y', \tau_{>i} Y) = 0$ for large i . This proves (2.18).

Since $Ho(f)$ is fully faithful we have

$$f_* f^! \xrightarrow{\sim} Id.$$

Let us check that every $M \in \mathbb{D}'$ the canonical morphism $M \rightarrow f^! f_* M$ is an isomorphism. Set $G = \text{cone}(M \rightarrow f^! f_* M)$. As we have just proved $G \in \mathbb{D}'(D_{dg}^b(\mathcal{A})^o \otimes_k D_{dg}^+(\mathcal{B}))$. On the other hand, the isomorphism $f_* f^! f_* \simeq f_*$ shows that $f_* G = 0$. Hence $G = 0$ by (2.14).

Next, consider the DG functor

$$g : D_{dg}^b(\mathcal{A})^o \otimes_k D_{dg}^b(\mathcal{B}) \rightarrow D_{dg}^+(\mathcal{A})^o \otimes_k D_{dg}^b(\mathcal{B})$$

and show that g^* and g_* define mutually inverse equivalences of categories

$$(2.20) \quad \mathbb{D}(D_{dg}^b(\mathcal{A})^o \otimes_k D_{dg}^b(\mathcal{B})) \simeq \mathbb{D}'',$$

where \mathbb{D}'' is a full subcategory of $\mathbb{D}(D_{dg}^+(\mathcal{A})^o \otimes_k D_{dg}^b(\mathcal{B}))$ whose objects are DG functors F satisfying property (2.15). Let us check that

$$(2.21) \quad g^*(\mathbb{D}(D_{dg}^b(\mathcal{A})^o \otimes_k D_{dg}^b(\mathcal{B}))) \subset \mathbb{D}''.$$

If $M \in \mathbb{D}(D_{dg}^b(\mathcal{A})^o \otimes_k D_{dg}^b(\mathcal{B}))$ is a functor representable by

$$Y \otimes Y' \in D_{dg}^b(\mathcal{A})^o \otimes_k D_{dg}^b(\mathcal{B})$$

then $g^* M$ is represented by the same object $Y \otimes Y'$ (viewed as an object of $D_{dg}^+(\mathcal{A})^o \otimes_k D_{dg}^b(\mathcal{B})$). Hence (2.16) is implied by the formula

$$\text{hocolim} \text{Hom}_{D_{dg}^+(\mathcal{A})}(Y, \tau_{<i} X) \simeq \text{Hom}_{D_{dg}^+(\mathcal{A})}(Y, X), \quad Y \in D_{dg}^b(\mathcal{A})$$

proved above (with \mathcal{A} replaced by \mathcal{B}). Since g^* commutes with arbitrary direct sums and since $\mathbb{D}(D_{dg}^b(\mathcal{A})^o \otimes_k D_{dg}^b(\mathcal{B}))$ is the smallest triangulated subcategory that contains representable functors and closed under direct sums $g^*(M) \in \mathbb{D}''(D_{dg}^+(\mathcal{A})^o \otimes_k D_{dg}^b(\mathcal{B}))$ for all M . By (2.15) the functor g_* is conservative when restricted to \mathbb{D}'' and the adjoint functor g^* is fully faithful (because $Ho(g)$ is fully faithful). It follows that $Id \xrightarrow{\sim} g_*g^*$ and $(g^*g_*)|_{\mathbb{D}''} \xrightarrow{\sim} Id$.

Combining (2.17) and (2.20) we see that the functors Res and $f^!g^*$ define mutually inverse equivalences between the category \mathbb{D}^f and $\mathbb{D}(D_{dg}^b(\mathcal{A})^o \otimes_k D_{dg}^b(\mathcal{B}))$. \square

Consider the composition

$$(2.22) \quad \mathbb{D}^f \xrightarrow{Res} \mathbb{D}(D_{dg}^b(\mathcal{A})^o \otimes_k D_{dg}^b(\mathcal{B})) \xrightarrow{\beta} D(PSh) \rightarrow D(Sh).$$

Combining Lemmas 2.1 and 2.8 we get the following.

Corollary 2.9. *Let $\mathbb{D}^{f+} \subset \mathbb{D}^f$ be the full subcategory whose objects are DG modules M such that $\beta \circ Res(M) \in D^+(PSh)$. Then (2.22) induces an equivalence of categories*

$$S : \mathbb{D}^{f+} \xrightarrow{\sim} D^+(Sh).$$

Lemma 2.10. *The functor $\mathcal{T} \hookrightarrow \mathbb{D}(D_{dg}^+(\mathcal{A})^o \otimes_k D_{dg}^+(\mathcal{B}))$ carries \mathcal{T}^+ into \mathbb{D}^{f+} .*

Proof. Let us show that every $\mathcal{F} \in \mathcal{T}$ satisfies property (2.6). By definition of \mathcal{T} , for every $X \in D_{dg}^+(\mathcal{A})$ there exists $Y \in D_{dg}^+(\mathcal{B})$ and an isomorphism

$$\mathcal{F}(X \times ?) \simeq Hom_{D_{dg}^+(\mathcal{B})}(?, Y)$$

in the derived category of right $D_{dg}^+(\mathcal{B})$ -modules. Property (2.6) follows because

$$Hom_{D_{dg}^+(\mathcal{B})}(\oplus_i X'^i, Y) \rightarrow \prod_i Hom_{D_{dg}^+(\mathcal{B})}(X'^i, Y).$$

is a quasi-isomorphism.

Let us show that every $\mathcal{F} \in \mathcal{T}^+$ satisfies property (2.7). Denote by $Ho(\mathcal{F}) : D_{dg}^+(\mathcal{A}) \rightarrow D^+(\mathcal{B})$ the triangulated functor associated with \mathcal{F} . By definition of $Ho(\mathcal{F})$ there is a functorial isomorphism

$$(2.23) \quad H^0(\mathcal{F}(X \otimes X')) \simeq Hom_{D^+(\mathcal{B})}(X', Ho\mathcal{F}(X))$$

In order to check (2.7) we will prove a stronger statement: for every $X' \in \mathcal{B}$ the morphism

$$(2.24) \quad Hom_{D^+(\mathcal{B})}(X', Ho\mathcal{F}(\tau_{<n}X)) \rightarrow Hom_{D^+(\mathcal{B})}(X', Ho\mathcal{F}(X))$$

is an isomorphism for sufficiently large n . By definition of \mathcal{T}^+

$$Ho\mathcal{F}(D^{>N}(\mathcal{A})) \subset D^{>0}(\mathcal{B}),$$

for some N . Then, for every $n > N$,

$$Ho\mathcal{F}(cone(\tau_{<n}X \rightarrow X)) \in D^{>0}(\mathcal{B})$$

and, hence,

$$Hom_{D^+(\mathcal{B})}(X', Ho\mathcal{F}(cone(\tau_{<n}X \rightarrow X))) = 0.$$

\square

Combining Lemma 2.10 and Corollary 2.9 we get a fully faithful embedding

$$(2.25) \quad S : \mathcal{T}^+ \hookrightarrow D(Sh).$$

By Lemma 2.4 S carries every quasi-functor \mathcal{F} satisfying property (P') to $s(H^0\mathcal{F}) \in Sh$. This proves the second part of Theorem 2. For the first part, let $F \in Fun(\mathcal{A}, \mathcal{B})$ be a k -linear functor, and let

$$(2.26) \quad "RF" \in \mathbb{D}(D_{dg}^+(\mathcal{A})^o \otimes_k D_{dg}^+(\mathcal{B}))$$

be the “derived functor” (see (2.11)). To complete the proof of Theorem it suffices to show the following.

Lemma 2.11. *Assume that F is left exact. Then “ RF ” $\in \mathbb{D}^{f+}$ and $S("RF") \xrightarrow{\sim} s(F)$.*

Proof. Let us show that “ RF ” satisfies property (2.14). According Remark 2.6 it will suffice to show that, for every integer n , $Y^i \in D_{dg}^{\geq n}(\mathcal{B})$ and $X \in HoC^+(\mathcal{A})$

$$H^0("RF"(X \otimes \bigoplus_i X'^i)) \xrightarrow{\sim} \prod_i H^0("RF"(X \otimes X'^i)).$$

We have ([Dri], §5)

$$(2.27) \quad H^0("RF"(X \otimes X')) \simeq \text{colim}_{Q_X} Hom_{D^+(\mathcal{B})}(X', F(Y)),$$

where Q_X is the filtrant category of pairs

$$(Y \in HoC_{dg}^+(\mathcal{A}), f \in Hom_{HoC_{dg}^+(\mathcal{A})}(X, Y))$$

such that $\text{cone}(f)$ is acyclic. If $X \in HoC^{\geq n}(\mathcal{A})$ the subcategory $Q'_X \subset Q_X$ formed by pairs (Y, f) with $Y \in HoC^{\geq n}(\mathcal{A})$ is cofinal in Q_X and, hence, Q_X in equation (2.27) can be replaced by Q'_X . Thus, it is enough to prove that the category Q_X has the following property: for every countable collection $w_i = (Y_i, f_i) \in Q'_X$, ($i = 1, 2, \dots$), there exists $v \in Q_X$ such that, for every i , the set $Mor_{Q_X}(w_i, v)$ is not empty. In fact, the object

$$v = (\text{cone}(\bigoplus_i X \xrightarrow{\phi} \bigoplus_i Y_i), g),$$

where $\phi_j : X \rightarrow \bigoplus_i Y_i$ equals $f_j - f_{j-1}$ and g is induced by the morphisms $X \xrightarrow{f_1} Y_1 \hookrightarrow \bigoplus_i Y_i$, does the job.

Let us show that “ RF ” satisfies property (2.15). As we explained in Remark 2.7 it suffices to show that

$$\text{colim } H^0("RF"(\tau_{<i} X \otimes X')) \xrightarrow{\sim} H^0("RF"(X \otimes X')),$$

for every $X' \in \mathcal{B}$. In fact, formula (2.27) with $Q_{\tau_{\geq i} X}$ replaced by $Q'_{\tau_{\geq i} X}$ shows that $H^0("RF"(\tau_{\geq i} X \otimes X')) = 0$ for $i > 0$. Hence, $H^0("RF"(\tau_{<i} X \otimes X')) \xrightarrow{\sim} H^0("RF"(X \otimes X'))$ is an isomorphism for $i > 1$. This proves that “ RF ” $\in \mathbb{D}^{f+}$.

For the second claim, observe that the restriction $Res("RF") \in \mathbb{D}(D_{dg}^b(\mathcal{A})^o \otimes_k D_{dg}^b(\mathcal{B}))$ is the bounded “derived functor” (2.11). Thus, we are done by Lemma 2.5. \square

Proof of theorem 3. Apply Corollary 2.3 and equation (2.25).

REFERENCES

- [BV] A. Beilinson, and V. Vologodsky, *A guide to Voevodsky's motives*. Geom. Funct. Anal. 17 (2008), no. 6, 1709-1787.
- [G] A. Grothendieck, *Sur quelques points d'algèbre homologique*. Tôhoku Math. J. (2) 9 (1957), 119–221.
- [Dri] V. Drinfeld, *DG quotients of DG categories* J. Algebra 272 (2004), no. 2, 643-691.
- [KS] M. Kashiwara, P. Schapira, *Categories and sheaves*, A series of comprehensive studies in Mathematics, v. 332, Springer.
- [K1] B. Keller, *On differential graded categories*, International Congress of Mathematicians. Vol. II, 151–190, Eur. Math. Soc., Zürich, 2006.
- [K2] B. Keller, *Deriving DG categories*, Ann. Sci. École Norm. Sup. (4) 27 (1994), no. 1, 63–102.
- [N] A. Neeman, *The derived category of an exact category*. J. Algebra 135 (1990), no. 2, 388-394.
- [Q] D. Quillen, *Higher algebraic K-theory: I*, Higher K-Theories, Lecture Notes in Mathematics, 341 (1972), Springer, pp. 85-147.
- [V2] V. Vologodsky, *The Albanese functor commutes with the Hodge realization*. arXiv:0809.2830 (2009).

ON THE DERIVED DG FUNCTORS

VADIM VOLOGODSKY

ABSTRACT. Assume that abelian categories \mathcal{A} , \mathcal{B} over a field admit countable direct limits and that these limits are exact. Let $\mathcal{F} : D_{dg}^+(\mathcal{A}) \rightarrow D_{dg}^+(\mathcal{B})$ be a DG quasi-functor such that the functor $Ho(\mathcal{F}) : D^+(\mathcal{A}) \rightarrow D^+(\mathcal{B})$ carries $D^{\geq 0}(\mathcal{A})$ to $D^{\geq 0}(\mathcal{B})$ and such that, for every $i > 0$, the functor $H^i \mathcal{F} : \mathcal{A} \rightarrow \mathcal{B}$ is effaceable. We prove that \mathcal{F} is canonically isomorphic to the right derived DG functor $RH^0(\mathcal{F})$. We also prove a similar result for bounded derived DG categories and a formula that expresses Hochschild cohomology of the categories $D_{dg}^b(\mathcal{A})$, $D_{dg}^+(\mathcal{A})$ as the *Ext* groups in the abelian category of left exact functors $\mathcal{A} \rightarrow Ind \mathcal{A}$. The proofs are based on a description of Drinfeld's category of quasi-functors as the derived category of a category of sheaves.

1. MAIN RESULTS

Let \mathcal{A} and \mathcal{B} be abelian categories, and let

$$RF_{tri} : D^+(\mathcal{A}) \rightarrow D^+(\mathcal{B})$$

be the right derived functor of some left exact functor $F : \mathcal{A} \rightarrow \mathcal{B}$. Then, the corresponding cohomological δ -functor $R^*F = H^*RF_{tri} : \mathcal{A} \rightarrow \mathcal{B}$ has the following property: the functor $H^i RF_{tri}$ is 0 for $i < 0$, effaceable for $i > 0$, and $H^0 RF_{tri}$ is isomorphic to F . Conversely, according to a result of Grothendieck ([G]), every cohomological δ -functor $T^* : \mathcal{A} \rightarrow \mathcal{B}$ satisfying the above property is canonically isomorphic to the right derived functor R^*F . The purpose of this paper is to extend this extremely useful characterization of R^*F to the derived category level. Unfortunately, Verdier's notion of triangulated functor seems too poor to allow such a simple characterization of the derived functors. In order to get a meaningful statement one has to consider triangulated functors with some kind of enrichment. Arguably the most useful notion here is the one of *DG quasi-functor* (or essentially equivalent notion of A_∞ -functor). Indeed, works of Keller and Drinfeld ([K2], [Dri]) provide a canonical DG enhancement $D_{dg}^+(\mathcal{A})$ of Verdier's triangulated derived category. Roughly, a DG quasi-functor $\mathcal{F} : D_{dg}^b(\mathcal{A}) \rightarrow D_{dg}^b(\mathcal{B})$ is a diagram of the form

$$(1.1) \quad D_{dg}^+(\mathcal{A}) \xleftarrow{S} \mathcal{C} \xrightarrow{G} D_{dg}^+(\mathcal{B}),$$

where \mathcal{C} is a DG category, S and G are DG functors, and, in addition, S is a homotopy equivalence. Every quasi-functor (1.1) yields a triangulated functor $Ho(\mathcal{F}) : D^+(\mathcal{A}) \rightarrow D^+(\mathcal{B})$, but the converse is not true in general. Nevertheless, many of the natural triangulated functors come together with a DG enhancement. For example, the triangulated derived functor RF can be canonically promoted to a DG quasi-functor ([Dri] §5). The main result of this paper states that under certain mild assumptions on abelian categories \mathcal{A} and \mathcal{B} the DG quasi-functors isomorphic

2000 *Mathematics Subject Classification*. Primary 13D09, 16E45, 18E25; Secondary 18E10.

Key words and phrases. Differential graded category, derived functor.

to the DG derived ones are precisely the DG quasi-functors satisfying Grothendieck's condition above. To state the result we need to introduce a bit of notation.

Let k be a commutative ring. Denote by $\text{Mod}(k)$ the category of k -modules. We shall say that a k -linear category ¹ is k -flat if, for every two objects X, Y , the k -module $\text{Hom}(X, Y)$ is flat. Given a k -linear exact category \mathcal{A} we denote by $D_{dg}^b(\mathcal{A})$ the corresponding bounded derived DG category over k . This is the DG quotient ([Dri]) of the DG category $C_{dg}^b(\mathcal{A})$ of bounded complexes by the subcategory of acyclic ones ([N], §1). The homotopy category of $D_{dg}^b(\mathcal{A})$ is the triangulated derived category $D^b(\mathcal{A})$ as defined in ([N]). Let \mathcal{B} be another k -linear abelian category, $D_{dg}^b(\mathcal{B})$ the corresponding bounded derived DG category, and let $\mathcal{T}(D_{dg}^b(\mathcal{A}), D_{dg}^b(\mathcal{B}))$ be the triangulated category of DG quasi-functors $\mathcal{F} : D_{dg}^b(\mathcal{A}) \rightarrow D_{dg}^b(\mathcal{B})$ ([Dri], §16.1). Given such \mathcal{F} and an integer i we denote by $H^i \mathcal{F} : \mathcal{A} \rightarrow \mathcal{B}$ the composition

$$\mathcal{A} \rightarrow D_{dg}^b(\mathcal{A}) \xrightarrow{\mathcal{F}} D_{dg}^b(\mathcal{B}) \xrightarrow{H^i} \mathcal{B}.$$

Theorem 1. *Let \mathcal{A} be a small k -flat exact idempotent complete category ² and \mathcal{B} a small abelian k -linear category.*

(1) *Assume that a DG quasi-functor*

$$\mathcal{F} : D_{dg}^b(\mathcal{A}) \rightarrow D_{dg}^b(\mathcal{B})$$

has the following property:

(P) *The functor $H^i \mathcal{F} : \mathcal{A} \rightarrow \mathcal{B}$ is 0 for every $i < 0$ and effaceable ³ for every $i > 0$.*

Then the functor $F := H^0 \mathcal{F} : \mathcal{A} \rightarrow \mathcal{B}$ is left exact, has a right derived DG quasi-functor ([Dri] §5)

$$RF : D_{dg}^b(\mathcal{A}) \rightarrow D_{dg}^b(\mathcal{B}),$$

and there is a unique isomorphism $\mathcal{F} \simeq RF$ such that the induced automorphism $F = H^0(\mathcal{F}) \simeq H^0(RF) = F$ equals Id. Conversely, the right derived DG quasi-functor of any left exact functor $F : \mathcal{A} \rightarrow \mathcal{B}$ satisfies property (P).

(2) *For every two DG quasi-functors $\mathcal{F}, \mathcal{G} \in \mathcal{T}(D_{dg}^b(\mathcal{A}), D_{dg}^b(\mathcal{B}))$ satisfying property (P) and every $i < 0$, we have*

$$\text{Hom}_{\mathcal{T}(D_{dg}^b(\mathcal{A}), D_{dg}^b(\mathcal{B}))}(\mathcal{F}, \mathcal{G}[i]) = 0,$$

$$\text{Hom}_{\mathcal{T}(D_{dg}^b(\mathcal{A}), D_{dg}^b(\mathcal{B}))}(\mathcal{F}, \mathcal{G}) = \text{Hom}_{\text{Fct}(\mathcal{A}, \mathcal{B})}(H^0 \mathcal{F}, H^0 \mathcal{G}).$$

Here $\text{Fct}(\mathcal{A}, \mathcal{B})$ denotes the category of all k -linear functors $\mathcal{A} \rightarrow \mathcal{B}$.

Remark 1.1. I do not know if the analogous statement holds for merely triangulated functors.

Remark 1.2. It is likely that the k -flatness assumption on \mathcal{A} is unnecessary. However, I can not prove this.

¹i.e., a category enriched over $\text{Mod}(k)$.

²An additive category is called idempotent complete if any its morphism $p : X \rightarrow X$ such that $p \circ p = p$ is the projection on a direct summand of a decomposition $X \simeq Y \oplus Z$.

³That is, for every object $X \in \mathcal{A}$, there exists an admissible monomorphism $X \hookrightarrow Y$ such that the induced morphism $H^i \mathcal{F}(X) \rightarrow H^i \mathcal{F}(Y)$ is 0.

We have a similar result for bounded from below derived DG categories. If \mathcal{A} is a k -linear abelian category we will write $D_{dg}^+(\mathcal{A})$ for the bounded from below derived DG category of \mathcal{A} and $D^+(\mathcal{A})$ for the corresponding triangulated category. Let $D^{\geq n}(\mathcal{A})$ be the full subcategory of $D^+(\mathcal{A})$ that consists of complexes with trivial cohomology in degrees less than n . We say that a DG quasi-functor

$$\mathcal{F} : D_{dg}^+(\mathcal{A}) \rightarrow D_{dg}^+(\mathcal{B})$$

has property (P') if

(P') The functor $Ho(\mathcal{F})$ takes every object of the category $D^{\geq 0}(\mathcal{A})$ to an object of $D^{\geq 0}(\mathcal{B})$ and, for every $i > 0$, the functor $H^i \mathcal{F} : \mathcal{A} \rightarrow \mathcal{B}$ is effaceable.

Theorem 2. *Let k be a field and let \mathcal{A}, \mathcal{B} be small abelian k -linear categories. Assume that both categories are closed under countable direct limits and that these limits are exact.*

- (1) *Let $\mathcal{F} \in \mathcal{T}(D_{dg}^+(\mathcal{A}), D_{dg}^+(\mathcal{B}))$ be a DG quasi-functor satisfying property (P') and $F := H^0 \mathcal{F} : \mathcal{A} \rightarrow \mathcal{B}$. The functor F admits a right derived DG quasi-functor $R\mathcal{F} : D_{dg}^+(\mathcal{A}) \rightarrow D_{dg}^+(\mathcal{B})$ and there is a unique isomorphism $\mathcal{F} \simeq R\mathcal{F}$ such that the induced automorphism $F = H^0(\mathcal{F}) \simeq H^0(R\mathcal{F}) = F$ equals Id . Conversely, a right derived DG quasi-functor of any left exact functor $F : \mathcal{A} \rightarrow \mathcal{B}$ satisfies property (P') .*
- (2) *For every two DG quasi-functors $\mathcal{F}, \mathcal{G} \in \mathcal{T}(D_{dg}^+(\mathcal{A}), D_{dg}^+(\mathcal{B}))$ satisfying property (P') and every $i < 0$, we have*

$$Hom_{\mathcal{T}(D_{dg}^+(\mathcal{A}), D_{dg}^+(\mathcal{B}))}(\mathcal{F}, \mathcal{G}[i]) = 0,$$

$$Hom_{\mathcal{T}(D_{dg}^+(\mathcal{A}), D_{dg}^+(\mathcal{B}))}(\mathcal{F}, \mathcal{G}) = Hom_{Fct(\mathcal{A}, \mathcal{B})}(H^0 \mathcal{F}, H^0 \mathcal{G}).$$

The main ingredient of the proof of Theorem 2 is the following construction. Let $Sh(\mathcal{A}^o \otimes_k \mathcal{B})$ be the category of k -linear contravariant functors $\mathcal{A}^o \otimes_k \mathcal{B} \rightarrow Mod(k)$ that are left exact with respect to both arguments. Every k -linear left exact functor $F : \mathcal{A} \rightarrow \mathcal{B}$ yields $s(F) \in Sh(\mathcal{A}^o \otimes_k \mathcal{B})$:

$$s(F)(X \otimes X') = Hom_{\mathcal{B}}(X', F(X)).$$

Let $\mathcal{T}^+ \subset \mathcal{T}(D_{dg}^+(\mathcal{A}), D_{dg}^+(\mathcal{B}))$ be the full triangulated subcategory whose objects are quasi-functors \mathcal{F} such that $Ho(\mathcal{F})(D^{\geq 0}(\mathcal{A})) \subset D^{\geq n}(\mathcal{B})$ for some n . Using key Lemma 2.1 we construct a fully faithful embedding

$$(1.2) \quad \mathcal{T}^+ \hookrightarrow D(Sh(\mathcal{A}^o \otimes_k \mathcal{B}))$$

that carries every DG quasi-functor \mathcal{F} satisfying property (P') to $s(F) \in Sh(\mathcal{A}^o \otimes_k \mathcal{B}) \subset D(Sh(\mathcal{A}^o \otimes_k \mathcal{B}))$.

Remark 1.3. In ([T], Th. 8.9), Toën gave an analogous description of the category of quasi-functors between the derived DG categories of (quasi)-coherent sheaves.

As another application of (1.2) we compute the Hochschild cohomology of a derived DG category. Recall (see, e.g. [K1], §5.4, [T], §8.1) that the Hochschild cohomology of a DG category \mathcal{C} can be interpreted as

$$(1.3) \quad HH^i(\mathcal{C}, \mathcal{C}) = Hom_{\mathcal{T}(\mathcal{C}, \mathcal{C})}(Id_{\mathcal{C}}, Id_{\mathcal{C}}[i]).$$

The composition in \mathcal{C} makes $HH^*(\mathcal{C}, \mathcal{C})$ a graded commutative algebra over k .

Theorem 3. *Let k be a field, and let \mathcal{A} be a small abelian k -linear category. There is an isomorphism of algebras*

$$(1.4) \quad HH^*(D_{dg}^b(\mathcal{A}), D_{dg}^b(\mathcal{A})) \simeq Ext_{Sh(\mathcal{A}^\circ \otimes_k \mathcal{A})}^*(s(Id_{\mathcal{A}}), s(Id_{\mathcal{A}})).$$

If, in addition, \mathcal{A} is closed under countable direct limits and that these limits are exact, we have

$$(1.5) \quad HH^*(D_{dg}^+(\mathcal{A}), D_{dg}^+(\mathcal{A})) \simeq Ext_{Sh(\mathcal{A}^\circ \otimes_k \mathcal{A})}^*(s(Id_{\mathcal{A}}), s(Id_{\mathcal{A}})).$$

Remark 1.4. This is a remarkable phenomenon the Hochschild cohomology does not change we “enlarge” the DG category. A similar result, that the Hochschild cohomology of a small DG category coincides with the Hochschild cohomology of its DG ind-completion, is due to Toën ([T], §8). An analogous statement for Grothendieck abelian categories was proved by Lowen and Van den Bergh ([LV]).

Remark 1.5. The category $Sh(\mathcal{A}^\circ \otimes_k \mathcal{A})$ has a tensor structure that extends the tensor structure on the category of left exact endofunctors $\mathcal{A} \rightarrow \mathcal{A}$ given by the composition. This can be used to promote (1.4), (1.5) to isomorphisms of *Gerstenhaber algebras* (see, e.g. [K1], §5.4).

Notation. Given a category \mathcal{C} we denote by \mathcal{C}° the opposite category. If \mathcal{C} is a DG category we will write $Ho\mathcal{C}$ for the corresponding homotopy category ([Dri], §2.7). For example, $Ho\mathcal{C}(Mod(k))$ denotes the homotopy category of complexes of k -modules. The derived category of right DG modules over a DG category \mathcal{C} will be denoted by $\mathbb{D}(\mathcal{C})$ ([Dri], §2.3)⁴. We will write $\underline{\mathcal{C}}$ for the DG category of semi-free right DG modules over \mathcal{C} ([BV], 1.6.1). We have a canonical equivalence of triangulated categories $Ho\underline{\mathcal{C}} \xrightarrow{\sim} \mathbb{D}(\mathcal{C})$ ([BV], 1.6.4). For DG categories $\mathcal{C}, \mathcal{C}'$ we denote by $\mathcal{T}(\mathcal{C}, \mathcal{C}')$ the category of DG quasi-functors ([Dri], §16.1). If \mathcal{C}' is a pretriangulated ([Dri], §2.4) $\mathcal{T}(\mathcal{C}, \mathcal{C}')$ has a canonical structure of triangulated category. If $\mathcal{F} \in \mathcal{T}(\mathcal{C}, \mathcal{C}')$ we will write $Ho(\mathcal{F})$ for the corresponding functor between the homotopy categories. The expression “direct limit” always means “filtrant direct limit” ([KS], §3).

Acknowledgements. I would like to thank Sasha Beilinson, Bernhard Keller, and Dima Orlov for helpful conversations related to the subject of this paper. My deep thanks due to the referee for his numerous remarks and for pointing out an error in a preliminary version of this paper. This research was partially supported by NSF grant DMS-0901707. Writing this paper I had in mind an application to the theory of Voevodsky’s motives ([V]). However, I believe that the main result explained here is interesting on its own ground.

2. PROOFS

Proof of theorem 1. Let $\mathcal{T}^+ \subset \mathcal{T} := \mathcal{T}(D_{dg}^b(\mathcal{A}), D_{dg}^b(\mathcal{B}))$ be the full triangulated subcategory whose objects are quasi-functors \mathcal{F} such that $H^i\mathcal{F} = 0$ for sufficiently small i . To prove the Theorem, we shall construct (in Lemma 2.1 below) a fully faithful embedding of \mathcal{T}^+ into the derived category of a certain abelian category $Sh(\mathcal{A}^\circ \otimes_k \mathcal{B})$ that takes every functor $\mathcal{F} \in \mathcal{T}^+$ satisfying property (P) to an object of the heart $Sh(\mathcal{A}^\circ \otimes_k \mathcal{B}) \subset D(Sh(\mathcal{A}^\circ \otimes_k \mathcal{B}))$.

Under our flatness assumption on \mathcal{A} , the category \mathcal{T} is a full subcategory of the derived category $\mathbb{D}(D_{dg}^b(\mathcal{A})^\circ \otimes_k D_{dg}^b(\mathcal{B}))$ of right DG modules over $D_{dg}^b(\mathcal{A})^\circ \otimes_k D_{dg}^b(\mathcal{B})$

⁴Drinfeld’s notation for this category is $D(\mathcal{C})$. We use a different notation to avoid a possible confusion with Verdier’s derived category of an abelian category \mathcal{C} that is denoted by $D(\mathcal{C})$.

that consists of all $M \in \mathbb{D}(D_{dg}^b(\mathcal{A})^o \otimes_k D_{dg}^b(\mathcal{B}))$ such that, for every X in $D_{dg}^b(\mathcal{A})^o$, the module $M(X) \in \mathbb{D}(D_{dg}^b(\mathcal{B}))$ belongs to the essential image of the Yoneda embedding $D_{dg}^+(\mathcal{B}) \rightarrow \mathbb{D}(D_{dg}^b(\mathcal{B}))$ ([Dri], §16.1).

Consider the restriction functor

$$\mathbb{D}(D_{dg}^b(\mathcal{A})^o \otimes_k D_{dg}^b(\mathcal{B})) \xrightarrow{\beta} \mathbb{D}(\mathcal{A}^o \otimes_k \mathcal{B})$$

induced by the DG quasi-functor $\mathcal{A}^o \otimes_k \mathcal{B} \rightarrow D_{dg}^b(\mathcal{A})^o \otimes_k D_{dg}^b(\mathcal{B})$. By definition, the triangulated category $\mathbb{D}(\mathcal{A}^o \otimes_k \mathcal{B})$ is the derived category of the abelian category $PSh := PSh(\mathcal{A}^o \otimes_k \mathcal{B})$ of k -linear presheaves *i.e.*, the category of k -linear contravariant functors $\mathcal{A}^o \otimes_k \mathcal{B} \rightarrow Mod(k)$. Consider a Grothendieck topology on $\mathcal{A}^o \otimes_k \mathcal{B}$ whose covers are maps of the form $f \otimes g : Y \otimes Y' \rightarrow X \otimes X'$, where $X, Y \in \mathcal{A}^o$, $X', Y' \in \mathcal{B}$, and $f : Y \rightarrow X$, $g : Y' \rightarrow X'$ are admissible epimorphisms⁵ *i.e.*, a sieve \mathcal{C} over $X \otimes X'$ is a covering sieve if there exist $f : Y \rightarrow X$, $g : Y' \rightarrow X'$ as above such that $Y \otimes Y' \xrightarrow{f \otimes g} X \otimes X' \in \mathcal{C}$. The axioms of Grothendieck topology (see, e.g. [KS], §16.1) are immediate except for the one which is the following statement: for every cover $Y \otimes Y' \xrightarrow{f \otimes g} X \otimes X'$ and every morphism $Z \otimes Z' \xrightarrow{\phi} X \otimes X'$ there exists a cover $T \otimes T' \xrightarrow{p \otimes q} Z \otimes Z'$ and a morphism $T \otimes T' \xrightarrow{\psi} Y \otimes Y'$ such that $(f \otimes g) \circ \psi = \phi \circ (p \otimes q)$, which is a consequence of the base change axiom of exact category ([Q], §2). Let $Sh := Sh(\mathcal{A}^o \otimes_k \mathcal{B})$ be the subcategory of PSh that consists of objects satisfying the sheaf property. Explicitly, objects of the category $Sh(\mathcal{A}^o \otimes_k \mathcal{B})$ are contravariant functors $\mathcal{A}^o \otimes_k \mathcal{B} \rightarrow Mod(k)$ that are left exact with respect to both arguments. The embedding $Sh \rightarrow PSh$ has a left adjoint functor (sheafification)

$$\sim : PSh \rightarrow Sh,$$

which is exact ([KS], §17.4). We denote by $\gamma : D(PSh) \rightarrow D(Sh)$ the induced functor between the derived categories. The composition

$$\mathbb{D}(D_{dg}^b(\mathcal{A})^o \otimes_k D_{dg}^b(\mathcal{B})) \xrightarrow{\beta} D(PSh) \xrightarrow{\gamma} D(Sh)$$

is *not* fully faithful in general, however, we have the following result.

Lemma 2.1. (cf. [T], Th. 8.9) *Let $\mathbb{D}^+ \subset \mathbb{D}(D_{dg}^b(\mathcal{A})^o \otimes_k D_{dg}^b(\mathcal{B}))$ be the full subcategory whose objects are DG modules M such that $\beta(M)$ is bounded from below. Then the functor*

$$S : \mathbb{D}^+ \xrightarrow{\beta} D^+(PSh) \xrightarrow{\gamma} D^+(Sh)$$

is an equivalence of categories.

Proof. The category $D_{dg}^b(\mathcal{A})^o \otimes_k D_{dg}^b(\mathcal{B})$ is the DG quotient of the category $C_{dg}^b(\mathcal{A})^o \otimes_k C_{dg}^b(\mathcal{B})$ by the full subcategory whose objects are of the form $X^\cdot \otimes X'^\cdot$, where either X^\cdot or X'^\cdot is acyclic. It then follows from ([Dri], Theorem 1.6.2) that the functor

$$\beta : \mathbb{D}(D_{dg}^b(\mathcal{A})^o \otimes_k D_{dg}^b(\mathcal{B})) \rightarrow \mathbb{D}(C_{dg}^b(\mathcal{A})^o \otimes_k C_{dg}^b(\mathcal{B})) = D(PSh)$$

is fully faithful and that its essential image consists of all DG-modules $M \in \mathbb{D}(C_{dg}^b(\mathcal{A})^o \otimes_k C_{dg}^b(\mathcal{B}))$ that carry every $X^\cdot \otimes X'^\cdot$ with the above property to an acyclic complex. Identifying the category $\mathbb{D}(C_{dg}^b(\mathcal{A})^o \otimes_k C_{dg}^b(\mathcal{B}))$ with $D(PSh)$ and observing that the subcategories of acyclic complexes in the homotopy categories $HoC_{dg}^b(\mathcal{A})$, $HoC_{dg}^b(\mathcal{B})$

⁵By definition, admissible epimorphisms $Y \rightarrow X$ in \mathcal{A}^o are admissible monomorphisms $X \rightarrow Y$ in \mathcal{A} .

are generated by short exact sequences ([N], §1) we exhibit $\mathbb{D}(D_{dg}^b(\mathcal{A})^o \otimes_k D_{dg}^b(\mathcal{B}))$ as a full subcategory $\mathcal{R} \subset D(PSh)$ whose objects are complexes F^\cdot of presheaves satisfying the following two conditions:

- For any exact sequence $0 \rightarrow Z \rightarrow Y \rightarrow X \rightarrow 0$ in \mathcal{A}^o and any $X' \in \mathcal{B}$ the total complex of

$$(2.1) \quad F^\cdot(X \otimes X') \rightarrow F^\cdot(Y \otimes X') \rightarrow F^\cdot(Z \otimes X')$$

is acyclic.

- For any $X \in \mathcal{A}^o$ and any exact sequence $0 \rightarrow Z' \rightarrow Y' \rightarrow X' \rightarrow 0$ in \mathcal{B} the total complex of

$$F^\cdot(X \otimes X') \rightarrow F^\cdot(X \otimes Y') \rightarrow F^\cdot(X \otimes Z')$$

is acyclic.

Observe that, for every $F^\cdot \in \mathcal{R}$ and an exact sequence $0 \rightarrow Z \rightarrow Y \rightarrow X \rightarrow 0$ in \mathcal{A}^o , we have a long exact sequence of k -modules

$$(2.2) \quad \cdots H^{m-1}(F^\cdot(Z \otimes X')) \rightarrow H^m(F^\cdot(X \otimes X')) \rightarrow H^m(F^\cdot(Y \otimes X')) \rightarrow H^m(F^\cdot(Z \otimes X')) \rightarrow \cdots$$

The equivalence of categories

$$\beta : \mathbb{D}(D_{dg}^b(\mathcal{A})^o \otimes_k D_{dg}^b(\mathcal{B})) \xrightarrow{\sim} \mathcal{R} \subset D(PSh)$$

carries \mathbb{D}^+ to the subcategory \mathcal{R}^+ of \mathcal{R} that consists of bounded from below complexes.

The derived category of sheaves $D(Sh)$ is the quotient of the derived category of presheaves by the subcategory $\mathcal{I}_{lac} \subset D(PSh)$ of locally (for our Grothendieck topology on $\mathcal{A}^o \otimes_k \mathcal{B}$) acyclic complexes ([BV], §1.11). We shall prove that

$$(2.3) \quad \mathcal{R}^+ \subset \mathcal{I}_{lac}^\perp,$$

where \mathcal{I}_{lac}^\perp denotes the right orthogonal complement to \mathcal{I}_{lac} in $D(PSh)$ ([BV] §1.1); *i.e.*

$$(2.4) \quad Hom_{D(PSh)}(G^\cdot, F^\cdot) = 0.$$

for every $G^\cdot \in \mathcal{I}_{lac}$ and $F^\cdot \in \mathcal{R}^+$. Without loss of generality we may assume that F^\cdot has trivial cohomology in negative degrees: $F^\cdot = F^0 \rightarrow F^1 \rightarrow \cdots$. Let $\tilde{F}^\cdot = \tilde{F}^0 \rightarrow \tilde{F}^1 \rightarrow \cdots$ be the corresponding complex of sheaves. Since the category of sheaves has enough injective objects (see, e.g. [KS], Th. 9.6.2, 18.1.6) there exists a complex $I^\cdot = I^0 \rightarrow I^1 \rightarrow \cdots$ of injective sheaves together with a morphism $\tilde{F}^\cdot \rightarrow I^\cdot$ which is an isomorphism in the derived category of sheaves. Let us show that the composition

$$\delta : F^\cdot \rightarrow \tilde{F}^\cdot \rightarrow I^\cdot$$

is an isomorphism in the derived category of presheaves. Indeed, every injective sheaf, viewed as a presheaf, is an object of \mathcal{R} . Thus I^\cdot and $cone(\delta)$ are in \mathcal{R}^+ . Assuming that $cone(\delta) \neq 0$ choose the smallest integer m such that

$$0 \neq H^m(cone(\delta)) \in PSh.$$

Then, there exist an object $X \otimes X' \in \mathcal{A}^o \otimes_k \mathcal{B}$ and a nonzero element $a \in H^m(cone(\delta))(X \otimes X')$. Since the sheafification of $H^m(cone(\delta))$ is 0 there exists a cover $p : Y \otimes Y' \rightarrow X \otimes X'$ such that

$$0 = p^* a \in H^m(cone(\delta))(Y \otimes Y').$$

Writing p as a composition

$$Y \otimes Y' \xrightarrow{1 \otimes q} Y \otimes X' \xrightarrow{f \otimes 1} X \otimes X'$$

we may assume $(f \otimes 1)^*a = 0$ (otherwise, we replace $X \otimes X'$ by $Y \otimes X'$). Let us look at the following fragment of the long exact sequence (2.2) applied to $F = \text{cone}(\delta)$ and the exact sequence $0 \rightarrow Z \rightarrow Y \xrightarrow{f} X \rightarrow 0$:

$$H^{m-1}(\text{cone}(\delta))(Z \otimes X') \rightarrow H^m(\text{cone}(\delta))(X \otimes X') \rightarrow H^m(\text{cone}(\delta))(Y \otimes X').$$

Since, by our assumption, $H^{m-1}(\text{cone}(\delta)) = 0$, it follows that $(f \otimes 1)^*$ is injective and, hence, $a = 0$. This contradiction proves that $\text{cone}(\delta) = 0$ i.e., δ is a quasi-isomorphism. Thus, to complete the proof of (2.4) it suffices to show that

$$\text{Hom}_{D(PSh)}(G', I') = 0,$$

for every $G' \in \mathcal{I}_{lac}$ and every bounded from below complex of injective sheaves I' . Indeed, every morphism $h : G' \rightarrow I'$ in the derived category is represented by a diagram in $C(PSh(\mathcal{A}^\circ \otimes_k \mathcal{B}))$

$$G' \leftarrow G'' \xrightarrow{h'} I',$$

where the first arrow is a quasi-isomorphism (and, in particular, $G'' \in \mathcal{I}_{lac}$). If h' is homotopic to 0 then h is 0 in the derived category. Thus, it is enough to show that

$$\text{Hom}_{K(PSh)}(G'', I') = 0,$$

where $K(PSh)$ denotes the homotopy category of complexes. We have

$$\text{Hom}_{K(PSh)}(G'', I') \xrightarrow{\sim} \text{Hom}_{K(Sh)}(\tilde{G}'', I') \xrightarrow{\sim} \text{Hom}_{D(Sh)}(\tilde{G}'', I').$$

The first arrow is an isomorphism because all terms of the complex I' are sheaves; the second arrow is an isomorphism by ([KS], Lemma 13.2.4). Finally, the group $\text{Hom}_{D(Sh)}(\tilde{G}'', I')$ is trivial because the sheafification \tilde{G}'' is 0 in $D(Sh)$.

To finish the proof of the lemma, we observe that, for every triangulated category \mathcal{C} and its full triangulated subcategory \mathcal{I} , the composition

$$\mathcal{I}^\perp \rightarrow \mathcal{C} \rightarrow \mathcal{C}/\mathcal{I}$$

is a fully faithful embedding: for every $X, Y \in \mathcal{C}$

$$\text{Hom}_{\mathcal{C}/\mathcal{I}}(X, Y) := \text{colim}_{f: X' \rightarrow X} \text{Hom}_{\mathcal{C}}(X', Y),$$

where the colimit is taken over the filtrant category of pairs $(X' \in \mathcal{C}, f : X' \rightarrow X)$ such that $\text{cone } f \in \mathcal{I}$. If $Y \in \mathcal{I}^\perp$, then

$$\text{Hom}_{\mathcal{C}}(X, Y) \xrightarrow{\sim} \text{Hom}_{\mathcal{C}}(X', Y),$$

and, hence,

$$\text{Hom}_{\mathcal{C}/\mathcal{I}}(X, Y) = \text{Hom}_{\mathcal{C}}(X, Y).$$

Applying this remark to $\mathcal{C} = D(PSh)$, $\mathcal{I} = \mathcal{I}_{lac}$ and using (2.4) we conclude that the functor $\mathcal{R}^+ \xrightarrow{\gamma} D(Sh)$ is fully faithful and, hence, so is the composition $\mathbb{D}^+ \xrightarrow{\sim} \mathcal{R}^+ \xrightarrow{\gamma} D(Sh)$. The essential image the functor $\mathcal{R}^+ \xrightarrow{\gamma} D(Sh)$ coincides with $D^+(Sh)$ because every complex of injective sheaves viewed as a complex of presheaves is an object of \mathcal{R}^+ . \square

Remark 2.2. Applying Lemma 2.1 to $k = \mathbb{Z}$ and \mathcal{A} being the category of free abelian groups of finite rank we obtain the following statement: for every small abelian category \mathcal{B}

$$\mathbb{D}^+(D_{dg}^b(\mathcal{B})) \xrightarrow{\sim} D^+(PSh(\mathcal{B})) = D^+(Ind(\mathcal{B})),$$

where $\mathbb{D}^+(D_{dg}^b(\mathcal{B}))$ is the full subcategory of $\mathbb{D}(D_{dg}^b(\mathcal{B}))$ that maps to $D^+(PSh(\mathcal{B}))$ under the restriction functor (and the ind-completion $Ind(\mathcal{B})$ is just another name for $PSh(\mathcal{B})$ ([KS], §8.6)). Note the functor

$$(2.5) \quad \mathbb{D}(D_{dg}^b(\mathcal{B})) \rightarrow D(Ind(\mathcal{B}))$$

is not an equivalence of categories in general. In fact, the functor (2.5) factors as

$$(2.6) \quad \mathbb{D}(D_{dg}^b(\mathcal{B})) \xrightarrow{\phi} HoC(Ind(\mathcal{B}))/Ho\overline{C_{ac}^b(\mathcal{B})} \xrightarrow{p} D(Ind(\mathcal{B})),$$

where $Ho\overline{C_{ac}^b(\mathcal{B})}$ is the smallest triangulated subcategory of the homotopy category of acyclic complexes $HoC_{ac}(Ind(\mathcal{B}))$ that contains *finite* acyclic complexes $HoC_{ac}^b(\mathcal{B})$ and closed under arbitrary direct sums; the functor p is the projection

$$HoC(Ind(\mathcal{B}))/Ho\overline{C_{ac}^b(\mathcal{B})} \rightarrow HoC(Ind(\mathcal{B}))/HoC_{ac}(Ind(\mathcal{B})).$$

The equivalence ϕ can be constructed as follows. Let $\overline{C_{ac}^b(\mathcal{B})}$ be the full subcategory of the DG category $C(Ind(\mathcal{B}))$ whose objects are those of $Ho\overline{C_{ac}^b(\mathcal{B})}$. The DG quasi-functor $D_{dg}^b(\mathcal{B}) \rightarrow C(Ind(\mathcal{B}))/\overline{C_{ac}^b(\mathcal{B})}$ extends uniquely to a quasi-functor

$$\phi_{dg} : D_{dg}^b(\mathcal{B}) \rightarrow C(Ind(\mathcal{B}))/\overline{C_{ac}^b(\mathcal{B})}$$

that commutes with arbitrary direct sums ([BV], §1.6.1). Define

$$\phi := Ho\phi_{dg}.$$

Let us show that ϕ is an equivalence of categories. The subcategory $Ho\overline{C_{ac}^b(\mathcal{B})} \subset HoC(Ind(\mathcal{B}))$ is generated by compact objects (e.g., objects of $HoC_{ac}^b(\mathcal{B})$); it follows that the projection $HoC(Ind(\mathcal{B})) \rightarrow HoC(Ind(\mathcal{B}))/Ho\overline{C_{ac}^b(\mathcal{B})}$ carries compact objects of $HoC(Ind(\mathcal{B}))$ to compact objects of the quotient category ([BV], §1.4.2). In particular, in the following commutative diagram

$$\begin{array}{ccc} D_{dg}^b(\mathcal{B}) & = & D_{dg}^b(\mathcal{B}) \\ \downarrow i & & \downarrow j \\ \mathbb{D}(D_{dg}^b(\mathcal{B})) & \xrightarrow{\phi} & HoC(Ind(\mathcal{B}))/Ho\overline{C_{ac}^b(\mathcal{B})} \end{array}$$

the image of j consists of compact objects. The same is true for the image of i ([BV], §1.7). The functors i, j are fully faithful and their images generate the categories $\mathbb{D}(D_{dg}^b(\mathcal{B}))$, $HoC(Ind(\mathcal{B}))/Ho\overline{C_{ac}^b(\mathcal{B})}$ respectfully. It follows that ϕ is an equivalence of categories.

In general, (e.g., if \mathcal{B} is the category of finitely generated modules over a finite group) the projection p is not conservative. However, if the category \mathcal{B} has *finite homological dimension* the objects of $D_{dg}^b(\mathcal{B})$ are compact in $D_{dg}^b(Ind(\mathcal{B}))$ ⁶ and the above argument proves that (2.5) is an equivalence of categories.

Corollary 2.3. *The composition*

$$(2.7) \quad S : \mathcal{T}^+ \xrightarrow{\alpha} \mathbb{D}(D_{dg}^b(\mathcal{A})^o \otimes_k D_{dg}^b(\mathcal{B})) \xrightarrow{\beta} D(PSh) \xrightarrow{\gamma} D(Sh)$$

is a fully faithful embedding.

⁶Indeed, under our finiteness assumption every complex in $D_{dg}^b(\mathcal{B})$ is quasi-isomorphic to a finite complex of projective objects. Thus it is enough to show that every projective object of \mathcal{B} is compact in $D(Ind(\mathcal{B}))$. This is clear because every such object is projective and compact in $Ind(\mathcal{B})$.

Consider the Yoneda embedding

$$s : \text{Fun}(\mathcal{A}, \mathcal{B}) \rightarrow \text{PSh}$$

that takes a functor $F \in \text{Fun}(\mathcal{A}, \mathcal{B})$ to the presheaf

$$s(F)(X \times X') = \text{Hom}_{\mathcal{B}}(X', F(X)).$$

If F is left exact then $s(F)$ is actually a sheaf.

Let $\mathcal{F} \in \mathcal{T}$ be a DG quasi-functor satisfying property (P). It follows from the definition of \mathcal{T}^+ given at the beginning of this section that $\mathcal{F} \in \mathcal{T}^+$. We shall prove that $S(\mathcal{F}) \xrightarrow{\sim} s(H^0\mathcal{F})$. Having in mind applications to Theorem 2 we will actually show a slightly more general statement. Namely, let us extend the functor (2.7) to a larger category:

$$S' : \mathcal{T}(D_{dg}^b(\mathcal{A}), D_{dg}^+(\mathcal{B})) \xrightarrow{\alpha'} \mathbb{D}(D_{dg}^b(\mathcal{A})^o \otimes_k D_{dg}^+(\mathcal{B})) \xrightarrow{\beta'} D(\text{PSh}) \xrightarrow{\gamma} D(\text{Sh}).$$

Lemma 2.4. *Let $\mathcal{F} \in \mathcal{T}(D_{dg}^b(\mathcal{A}), D_{dg}^+(\mathcal{B}))$ be a DG quasi-functor such that $H^i\mathcal{F}$ is zero for $i < 0$ and effaceable for $i > 0$. Set $s(F) = s(H^0\mathcal{F}) \subset \text{Sh} \subset D(\text{Sh})$ ⁷. Then the complex $S'(\mathcal{F}) \in D(\text{Sh})$ is canonically quasi-isomorphic to $s(F)$.*

Proof. By definition, the cohomology presheaves of the complex $\beta'\alpha'(\mathcal{F}) \in D(\text{PSh})$ are given by the formula

$$H^i(\beta'\alpha'\mathcal{F})(X \otimes X') = \text{Hom}_{D^+(\mathcal{B})}(X', \text{Ho}(\mathcal{F})(X)[i]).$$

Since the negative cohomology of the complex $\text{Ho}(\mathcal{F})(X) \in D^+(\mathcal{B})$ vanishes the same is true for $\beta'\alpha'\mathcal{F}$ and, thus, we have

$$H^0(\beta'\alpha'\mathcal{F})(X \otimes X') = \text{Hom}_{D^+(\mathcal{B})}(X', H^0\mathcal{F}(X)) = s(F).$$

It remains to prove that for every $i > 0$ the sheafification of the presheaf $H^i(\beta'\alpha'\mathcal{F})$ equals zero. Given an integer j define presheaves $G^{i,j}$ to be

$$G^{i,j}(X \otimes X') = \text{Hom}_{D^+(\mathcal{B})}(X', \tau_{\leq j}(\text{Ho}(\mathcal{F})(X)))[i].$$

We shall show by induction on j that for every $i > 0$ and every j the sheafification of $G^{i,j}$ is 0. This would complete the proof since $G^{i,j}$ is isomorphic to $H^i(\beta'\alpha'\mathcal{F})(X \otimes X')$ for $j \geq i$. For every $i > 0$ and every element v of the group

$$G^{i,0}(X \otimes X') = \text{Ext}_{\mathcal{B}}^i(X', H^0\mathcal{F}(X))$$

there exists an epimorphism $Y' \rightarrow X'$ such that v is annihilated by the map

$$\text{Ext}_{\mathcal{B}}^i(X', H^0\mathcal{F}(X)) \rightarrow \text{Ext}_{\mathcal{B}}^i(Y', H^0\mathcal{F}(X))$$

([KS], Exercise 13.17). This proves that the sheafification of $G^{i,0}$ is 0. For the induction step, consider the distinguished triangle

$$\tau_{\leq j}(\text{Ho}(\mathcal{F})(X)) \rightarrow \tau_{\leq j+1}(\text{Ho}(\mathcal{F})(X)) \rightarrow H^{j+1}\mathcal{F}(X)[-j-1]$$

and the corresponding long exact sequence

$$\rightarrow G^{i,j}(X \otimes X') \rightarrow G^{i,j+1}(X \otimes X') \rightarrow \text{Hom}_{D^b(\mathcal{B})}(X', H^{j+1}\mathcal{F}(X)[-j-1+i]) \rightarrow .$$

It follows that $G^{i,j+1}$ fits in a long exact sequence

$$\rightarrow G^{i,j} \rightarrow G^{i,j+1} \rightarrow \text{Ext}_{\mathcal{B}}^{i-j-1}(\cdot, H^{j+1}\mathcal{F}(\cdot)) \rightarrow .$$

⁷The vanishing of $H^i\mathcal{F}$ implies that F is left exact and, hence, $s(F)$ is a sheaf.

The sheafification of $G^{i,j}$ is 0 by the induction assumption, the sheafification of $Ext_{\mathcal{B}}^{i-j-1}(\cdot, H^{j+1}\mathcal{F}(\cdot))$ is 0 because the functor $H^{j+1}\mathcal{F}$ is effaceable. Hence, the sheafification of $G^{i,j+1}$ is 0 as well. \square

Now we are ready to prove the second part of the theorem. Given quasi-functors $\mathcal{F}, \mathcal{G} \in \mathcal{T}$ satisfying property (P) we have by Lemmas 2.1, 2.4

$$(2.8) \quad Hom_{\mathcal{T}}(\mathcal{F}, \mathcal{G}[i]) \xrightarrow{\sim} Hom_{D(Sh)}(S(\mathcal{F}), S(\mathcal{G})[i]) \xrightarrow{\sim} Ext_{Sh}^i(s(H^0\mathcal{F}), s(H^0\mathcal{G})).$$

In particular, $Hom_{\mathcal{T}}(\mathcal{F}, \mathcal{G}[i])$ is isomorphic to $Hom_{Fun(\mathcal{A}, \mathcal{B})}(H^0\mathcal{F}, H^0\mathcal{G})$ for $i = 0$ (since the functor $s : Fun(\mathcal{A}, \mathcal{B}) \rightarrow PSh$ is fully faithful) and to 0 for $i < 0$.

To prove the first part of the theorem we need to recall some facts about DG categories and derived functors. Let $f : \mathcal{C}_1 \rightarrow \mathcal{C}_2$ be a DG functor between small DG categories. Then the restriction functor $f_* : \mathbb{D}(\mathcal{C}_2) \rightarrow \mathbb{D}(\mathcal{C}_1)$ admits a left and a right adjoint functors (the derived induction and co-induction functors)

$$(2.9) \quad f^*, f^! : \mathbb{D}(\mathcal{C}_1) \rightarrow \mathbb{D}(\mathcal{C}_2)$$

([Dri], §14.12). In particular, we have the canonical morphisms

$$(2.10) \quad \begin{aligned} Id &\rightarrow f_*f^*, & f_*f^! &\rightarrow Id \\ Id &\rightarrow f^!f_*, & f^*f_* &\rightarrow Id. \end{aligned}$$

It also follows from the adjunction property that f^* commutes with arbitrary direct sums and that $f^!$ commutes with arbitrary direct products. If the the functor $Ho(f) : Ho(\mathcal{C}_1) \rightarrow Ho(\mathcal{C}_2)$ is fully faithful so is f_* and the first two morphisms in (2.10) are isomorphisms.

Recall the definition of the derived DG quasi-functor RF of a left exact functor $F : \mathcal{A} \rightarrow \mathcal{B}$ from ([Dri], §16). Consider the functor

$$\mathcal{T}(\mathcal{A}, D_{dg}^b(\mathcal{B})) \hookrightarrow \mathbb{D}(C_{dg}^b(\mathcal{A})^o \otimes_k D_{dg}^b(\mathcal{B})) \xrightarrow{f^*} \mathbb{D}(D_{dg}^b(\mathcal{A})^o \otimes_k D_{dg}^b(\mathcal{B}))$$

induced by the projection

$$f : C_{dg}^b(\mathcal{A})^o \otimes_k D_{dg}^b(\mathcal{B}) \rightarrow D_{dg}^b(\mathcal{A})^o \otimes_k D_{dg}^+(\mathcal{B}).$$

Given a k -linear functor $F \in Fun(\mathcal{A}, \mathcal{B}) \rightarrow \mathcal{T}(\mathcal{A}, D_{dg}^b(\mathcal{B}))$ we define the “derived functor”

$$(2.11) \quad “RF” = f^*(F) \in \mathbb{D}(D_{dg}^b(\mathcal{A})^{op} \otimes_k D_{dg}^b(\mathcal{B})).$$

The right derived DG quasi-functor $RF : D_{dg}^b(\mathcal{A}) \rightarrow D_{dg}^b(\mathcal{B})$, if it exists, is an object of $\mathcal{T}(D_{dg}^b(\mathcal{A}), D_{dg}^b(\mathcal{B}))$ whose image in $\mathbb{D}(D_{dg}^b(\mathcal{A})^o \otimes_k D_{dg}^b(\mathcal{B})) \supset \mathcal{T}(D_{dg}^b(\mathcal{A}), D_{dg}^b(\mathcal{B}))$ is “ RF ”.

Lemma 2.5. *Assume that F is left exact. Then “ RF ” $\in \mathbb{D}^+ \subset \mathbb{D}(D_{dg}^b(\mathcal{A})^{op} \otimes_k D_{dg}^b(\mathcal{B}))$ and the functor $S : \mathbb{D}^+ \hookrightarrow D(Sh)$ takes “ RF ” to $s(F)$.*

Proof. Let $\beta : \mathbb{D}(D_{dg}^b(\mathcal{A})^o \otimes_k D_{dg}^b(\mathcal{B})) \rightarrow D(PSh)$ be the restriction functor, and let $\gamma : D(PSh) \rightarrow D(Sh)$ be the sheafification functor. As explained in ([Dri], §5) the presheaves $H^i(\beta(“RF”))$ can be computed as follows:

$$(2.12) \quad H^i(\beta(“RF”))(X \otimes X') = colim_Q Hom_{D^b(\mathcal{B})}(X', F(Y)[i]),$$

where the colimit is taken over the filtrant category Q of pairs $(Y, f) \in HoC_{dg}^b(\mathcal{A})$, $f \in Hom_{HoC_{dg}^b(\mathcal{A})}(X, Y)$ such that $cone(f)$ is acyclic. As the subcategory $Q' \subset Q$ consisting of pairs (Y, f) with $Y^j = 0$ for $j < 0$ is cofinal in Q , the category Q in the

equation (2.12) can be replaced by Q' . This proves that “ RF ” $\in \mathbb{D}^+$. Let us show that $\gamma \circ \beta(\text{“}RF\text{”}) \simeq s(F)$. We have

$$H^0(\beta(\text{“}RF\text{”}))(X \otimes X') = \text{colim}_{Q'} \text{Hom}_{D^b(\mathcal{B})}(X', F(Y')) \simeq$$

$$\text{colim}_{Q'} \text{Hom}_{D^b(\mathcal{B})}(X', \tau_{\leq 0} F(Y')) \simeq \text{colim}_{Q'} \text{Hom}_{D^b(\mathcal{B})}(X', F(X)) = s(F)(X \otimes X').$$

It remains to prove that, for every $i > 0$, the sheafification of $H^i(\beta(\text{“}RF\text{”}))$ is 0. Let s be the section of $H^i(\beta(\text{“}RF\text{”}))(X \otimes X')$ represented by an element

$$\tilde{s} \in \text{Hom}_{D^b(\mathcal{B})}(X', F(Y')[i]),$$

where $X \xrightarrow{f} Y^0 \rightarrow Y^1 \rightarrow \dots$ is an object of Q' . Looking at the diagram

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y^0 & \rightarrow & Y^1 & \rightarrow & \dots \\ \downarrow f & & \downarrow Id & & \downarrow & & \\ Y^0 & \xrightarrow{Id} & Y^0 & \rightarrow & 0 & \rightarrow & \dots \end{array}$$

we see that the pullback $(f \otimes Id)^* s \in H^i(\beta(\text{“}RF\text{”}))(Y^0 \otimes X')$ is represented by an element of the group $\text{Hom}_{D^+(\mathcal{B})}(X', F(Y^0)[i]) = \text{Ext}_{\mathcal{B}}^i(X', F(Y^0))$. For any positive i every element of this group is annihilated by the map $\text{Ext}_{\mathcal{B}}^i(X', F(Y^0)) \rightarrow \text{Ext}_{\mathcal{B}}^i(Y', F(Y^0))$ for some epimorphism $Y' \rightarrow X'$. \square

Let us prove the first part of the theorem. Let $\mathcal{F} \in \mathcal{T} \subset \mathbb{D}(D_{dg}^b(\mathcal{A})^o \otimes_k D_{dg}^b(\mathcal{B}))$ be a DG quasi-functor satisfying property (P) together with an isomorphism $F \simeq H^0 \mathcal{F}$. We need to construct an isomorphism $\mathcal{F} \simeq \text{“}RF\text{”}$. By Lemmas 2.4, 2.5 $\mathcal{F}, \text{“}RF\text{”}$ are objects of \mathbb{D}^+ . By Lemma 2.1 the functor $S : \mathbb{D}^+ \rightarrow D(Sh)$ is fully faithful. Thus, constructing an isomorphism $\mathcal{F} \simeq \text{“}RF\text{”}$ is equivalent to producing an isomorphism $S(\mathcal{F}) \simeq S(\text{“}RF\text{”})$ in $D(Sh)$ which was done in Lemmas 2.4, 2.5. Theorem 1 is proved.

Proof of theorem 2. Let $\mathcal{T}^+ \subset \mathcal{T} := \mathcal{T}(D_{dg}^+(\mathcal{A}), D_{dg}^+(\mathcal{B}))$ be the full triangulated subcategory whose objects are quasi-functors \mathcal{F} such that, for some integer n , we have

$$Ho(\mathcal{F})(D^{\geq 0}(\mathcal{A})) \subset D^{\geq n}(\mathcal{B}).$$

We shall prove that the composition

$$\mathcal{T}^+ \hookrightarrow \mathbb{D}(D_{dg}^+(\mathcal{A})^o \otimes_k D_{dg}^+(\mathcal{B})) \xrightarrow{Res} \mathbb{D}(D_{dg}^b(\mathcal{A})^o \otimes_k D_{dg}^b(\mathcal{B})) \rightarrow D(Sh)$$

is a fully faithful embedding. Here Res denotes the restriction functor induced by the embedding

$$(2.13) \quad D_{dg}^b(\mathcal{A})^o \otimes_k D_{dg}^b(\mathcal{B}) \rightarrow D_{dg}^+(\mathcal{A})^o \otimes_k D_{dg}^+(\mathcal{B}).$$

To show this we need to introduce a bit of notation. If \mathcal{C} is an abelian category closed under countable direct sums and

$$X^0 \xrightarrow{\phi_0} X^1 \xrightarrow{\phi_1} X^2 \xrightarrow{\phi_2} \dots$$

is a diagram of complexes $X^i \in C(\mathcal{C})$, we set

$$\text{hocolim } X^i = \text{cone}(\bigoplus_i X^i \xrightarrow{v} \bigoplus_i X^i) \in C(\mathcal{C}),$$

where $v|_{X^i} := Id_{X^i} - \phi_i : X^i \rightarrow \bigoplus_i X^i$. There is a canonical morphism

$$\text{hocolim } X^i \rightarrow \text{colim } X^i,$$

which is a quasi-isomorphism if countable direct limits in \mathcal{C} are exact. If this is the case, every morphism $X \rightarrow X'$ of diagrams that is a term-wise quasi-isomorphism induces a quasi-isomorphism of the homotopy colimits⁸. Dually, for a category \mathcal{C} closed under countable products and a diagram

$$\cdots \rightarrow X_2 \xrightarrow{\phi_1} X_1 \xrightarrow{\phi_0} X_0,$$

we set

$$\operatorname{holim} X_i = \operatorname{cone}\left(\prod_i X_i \xrightarrow{v} \prod_i X_i\right)[-1],$$

where $v_i := p_i - \phi_i p_{i+1} : \prod X_i \rightarrow X_i$ and $p_i : \prod X_i \rightarrow X_i$ are the projections.

Let $\mathbb{D}^f \subset \mathbb{D}(D_{dg}^+(\mathcal{A})^o \otimes_k D_{dg}^+(\mathcal{B}))$ be the full subcategory whose objects are the covariant DG functors $M : D_{dg}^+(\mathcal{A}) \otimes_k D_{dg}^+(\mathcal{B})^o \rightarrow C(\operatorname{Mod}(k))$ such that, for every $X \in D_{dg}^+(\mathcal{A})$ and $X' \in D_{dg}^+(\mathcal{B})$, the canonical morphism

$$(2.14) \quad M(X \otimes X') \rightarrow \operatorname{holim} M(X \otimes \tau_{<i} X'),$$

is a quasi-isomorphism, and, for every $X \in D_{dg}^+(\mathcal{A})$ and every *bounded* $X' \in D_{dg}^b(\mathcal{B})$, the canonical morphism

$$(2.15) \quad \operatorname{hocolim} M(\tau_{<i} X \otimes X') \rightarrow M(X \otimes X'),$$

is a quasi-isomorphism.

Remark 2.6. Since countable direct limits are exact in \mathcal{B} , the morphism $\operatorname{hocolim} \tau_{<i} X' \rightarrow X'$ is a quasi-isomorphism. Thus, property (2.14) is implied by the following: for every integer n and a countable collection $X^{n_i} \in D_{dg}^{\geq n}(\mathcal{B})$, the morphism

$$M(X \otimes \oplus_i X^{n_i}) \rightarrow \prod_i M(X \otimes X^{n_i})$$

is a quasi-isomorphism.

Remark 2.7. Since directed limits are exact in $\operatorname{Mod}(k)$ property (2.15) is equivalent to the following: for every $X \in D_{dg}^+(\mathcal{A})$ and $X' \in \mathcal{B}$, we have

$$(2.16) \quad \operatorname{colim} H^0(M(\tau_{<i} X \otimes X')) \xrightarrow{\sim} H^0(M(X \otimes X')).$$

Lemma 2.8. *The restriction functor*

$$\mathbb{D}^f \xrightarrow{\operatorname{Res}} \mathbb{D}(D_{dg}^b(\mathcal{A})^o \otimes_k D_{dg}^b(\mathcal{B}))$$

is an equivalence of categories.

Proof. We shall first consider the restriction

$$f_* : \mathbb{D}(D_{dg}^+(\mathcal{A})^o \otimes_k D_{dg}^+(\mathcal{B})) \rightarrow \mathbb{D}(D_{dg}^+(\mathcal{A})^o \otimes_k D_{dg}^b(\mathcal{B}))$$

and prove that $f^!$ and f_* define mutually inverse equivalences of categories

$$(2.17) \quad \mathbb{D}(D_{dg}^+(\mathcal{A})^o \otimes_k D_{dg}^b(\mathcal{B})) \simeq \mathbb{D}',$$

where \mathbb{D}' is the full subcategory of $\mathbb{D}(D_{dg}^+(\mathcal{A})^o \otimes_k D_{dg}^b(\mathcal{B}))$ whose objects are DG functors M satisfying the property (2.14). Let us check that

$$(2.18) \quad f^!(\mathbb{D}(D_{dg}^+(\mathcal{A})^o \otimes_k D_{dg}^b(\mathcal{B}))) \subset \mathbb{D}'.$$

⁸For the last property, it suffices to assume that countable direct sums are exact in \mathcal{C} .

For every DG functor $f : \mathcal{C}_1 \rightarrow \mathcal{C}_2$ between DG categories over a field, the functor $f^! : \mathbb{D}(\mathcal{C}_1) \rightarrow \mathbb{D}(\mathcal{C}_2)$ admits the following concrete description: if $M : \mathcal{C}_1 \rightarrow C(\text{Mod}(k))$ is a contravariant DG functor and X is an object of \mathcal{C}_2 , we have

$$(2.19) \quad f^!(M)(X) = \text{Hom}_{\mathbb{D}_{dg}(\mathcal{C}_1)}(f_*^{dg} \text{Hom}_{\mathcal{C}_2}(\cdot, X), M).$$

Here $\mathbb{D}_{dg}(\mathcal{C}_i)$ denotes the DG derived category of right \mathcal{C}_i -modules, f_*^{dg} the derived restriction functor, and $\text{Hom}_{\mathcal{C}_2}(\cdot, X)$ is the image of X under the Yoneda embedding $\mathcal{C}_2 \rightarrow \mathbb{D}_{dg}(\mathcal{C}_2)$.

We shall prove that

$$\text{hocolim} \text{Hom}(\cdot, X \otimes \tau_{<i} X') \rightarrow f_* \text{Hom}(\cdot, X \otimes X')$$

is an isomorphism in $\mathbb{D}(D_{dg}^+(\mathcal{A})^\circ \otimes_k D_{dg}^b(\mathcal{B}))$. Together with (2.19) it will imply (2.18). By definition of the tensor product of DG categories, for every $Y \otimes Y' \in D_{dg}^+(\mathcal{A})^\circ \otimes_k D_{dg}^b(\mathcal{B})$,

$$\text{Hom}(Y \otimes Y', X \otimes X') = \text{Hom}(Y, X) \otimes_k \text{Hom}(Y', X').$$

Hence, it is enough to check that the morphism

$$\text{hocolim} \text{Hom}_{D_{dg}^+(\mathcal{B})}(Y', \tau_{<i} Y) \rightarrow \text{Hom}_{D_{dg}^+(\mathcal{B})}(Y', Y)$$

is a quasi-isomorphism, for every $Y' \in D_{dg}^b(\mathcal{B})$. Using the exactness of direct limits in $\text{Mod}(k)$ the last assertion is reduced to the formula

$$\text{colim} \text{Hom}_{D^b(\mathcal{B})^\circ}(Y', \tau_{<i} Y) \simeq \text{Hom}_{D^+(\mathcal{B})^\circ}(Y', Y),$$

which holds because the group $\text{Hom}_{D^+(\mathcal{B})^\circ}(Y', \tau_{>i} Y)$ is trivial for large i . This proves the assertion (2.18).

Since the functor $Ho(f)$ is fully faithful, we have

$$f_* f^! \xrightarrow{\sim} Id.$$

Let us check that for every $M \in \mathbb{D}'$ the canonical morphism $M \rightarrow f^! f_* M$ is an isomorphism. Set $G = \text{cone}(M \rightarrow f^! f_* M)$. As we have just proved G belongs to $\mathbb{D}'(D_{dg}^b(\mathcal{A})^\circ \otimes_k D_{dg}^+(\mathcal{B}))$. On the other hand, the isomorphism $f_* f^! f_* \simeq f_*$ shows that $f_* G$ is 0. Hence, G is 0 by (2.14).

Next, consider the DG functor

$$g : D_{dg}^b(\mathcal{A})^\circ \otimes_k D_{dg}^b(\mathcal{B}) \rightarrow D_{dg}^+(\mathcal{A})^\circ \otimes_k D_{dg}^b(\mathcal{B})$$

and show that g^* and g_* define mutually inverse equivalences of categories

$$(2.20) \quad \mathbb{D}(D_{dg}^b(\mathcal{A})^\circ \otimes_k D_{dg}^b(\mathcal{B})) \simeq \mathbb{D}'' ,$$

where \mathbb{D}'' is a full subcategory of $\mathbb{D}(D_{dg}^+(\mathcal{A})^\circ \otimes_k D_{dg}^b(\mathcal{B}))$ whose objects are DG functors F satisfying property (2.15). Let us check that

$$(2.21) \quad g^*(\mathbb{D}(D_{dg}^b(\mathcal{A})^\circ \otimes_k D_{dg}^b(\mathcal{B}))) \subset \mathbb{D}'' .$$

If $M \in \mathbb{D}(D_{dg}^b(\mathcal{A})^\circ \otimes_k D_{dg}^b(\mathcal{B}))$ is a functor representable by

$$Y \otimes Y' \in D_{dg}^b(\mathcal{A})^\circ \otimes_k D_{dg}^b(\mathcal{B})$$

then $g^* M$ is represented by the same object $Y \otimes Y'$ (viewed as an object of $D_{dg}^+(\mathcal{A})^\circ \otimes_k D_{dg}^b(\mathcal{B})$). Hence (2.16) is implied by the formula

$$\text{hocolim} \text{Hom}_{D_{dg}^+(\mathcal{A})}(Y, \tau_{<i} X) \simeq \text{Hom}_{D_{dg}^+(\mathcal{A})}(Y, X), \quad Y \in D_{dg}^b(\mathcal{A})$$

proved above (with \mathcal{A} replaced by \mathcal{B}). Since g^* commutes with arbitrary direct sums and since $\mathbb{D}(D_{dg}^b(\mathcal{A})^\circ \otimes_k D_{dg}^b(\mathcal{B}))$ is the smallest triangulated subcategory that contains representable functors and closed under direct sums, $g^*(M)$ is an object of $\mathbb{D}''(D_{dg}^+(\mathcal{A})^\circ \otimes_k D_{dg}^b(\mathcal{B}))$ for every M . By (2.15) the functor g_* is conservative when restricted to \mathbb{D}'' and the adjoint functor g^* is fully faithful (because $Ho(g)$ is fully faithful). Hence, we have

$$Id \xrightarrow{\sim} g_* g^*, \quad (g^* g_*)|_{\mathbb{D}''} \xrightarrow{\sim} Id.$$

Combining equations (2.17) and (2.20) we see that the functors Res and $f^! g^*$ define mutually inverse equivalences between the category \mathbb{D}^f and the category $\mathbb{D}(D_{dg}^b(\mathcal{A})^\circ \otimes_k D_{dg}^b(\mathcal{B}))$. \square

Consider the composition

$$(2.22) \quad \mathbb{D}^f \xrightarrow{Res} \mathbb{D}(D_{dg}^b(\mathcal{A})^\circ \otimes_k D_{dg}^b(\mathcal{B})) \xrightarrow{\beta} D(PSh) \rightarrow D(Sh).$$

Combining Lemmas 2.1 and 2.8 we get the following.

Corollary 2.9. *Let $\mathbb{D}^{f+} \subset \mathbb{D}^f$ be the full subcategory whose objects are DG modules M such that $\beta \circ Res(M)$ is bounded from below. Then (2.22) induces an equivalence of categories*

$$S : \mathbb{D}^{f+} \xrightarrow{\sim} D^+(Sh).$$

Lemma 2.10. *The functor $\mathcal{T} \hookrightarrow \mathbb{D}(D_{dg}^+(\mathcal{A})^\circ \otimes_k D_{dg}^+(\mathcal{B}))$ carries \mathcal{T}^+ into \mathbb{D}^{f+} .*

Proof. Let us show that every $\mathcal{F} \in \mathcal{T}$ satisfies property (2.6). By definition of \mathcal{T} , for every $X \in D_{dg}^+(\mathcal{A})$, there exists $Y \in D_{dg}^+(\mathcal{B})$ and an isomorphism

$$\mathcal{F}(X \times ?) \simeq Hom_{D_{dg}^+(\mathcal{B})}(?, Y)$$

in the derived category of right $D_{dg}^+(\mathcal{B})$ -modules. Property (2.6) follows because the morphism

$$Hom_{D_{dg}^+(\mathcal{B})}(\oplus_i X^i, Y) \rightarrow \prod_i Hom_{D_{dg}^+(\mathcal{B})}(X^i, Y).$$

is a quasi-isomorphism.

Let us show that every $\mathcal{F} \in \mathcal{T}^+$ satisfies the property (2.7). Denote by $Ho(\mathcal{F}) : D_{dg}^+(\mathcal{A}) \rightarrow D^+(\mathcal{B})$ the triangulated functor associated with \mathcal{F} . By definition of $Ho(\mathcal{F})$ there is a functorial isomorphism

$$(2.23) \quad H^0(\mathcal{F}(X \otimes X')) \simeq Hom_{D^+(\mathcal{B})}(X', Ho\mathcal{F}(X))$$

In order to check (2.7) we will prove a stronger statement: for every $X' \in \mathcal{B}$ the morphism

$$(2.24) \quad Hom_{D^+(\mathcal{B})}(X', Ho\mathcal{F}(\tau_{<n} X)) \rightarrow Hom_{D^+(\mathcal{B})}(X', Ho\mathcal{F}(X))$$

is an isomorphism for sufficiently large n . By definition of \mathcal{T}^+ we can find an integer N such that the functor $Ho\mathcal{F}$ carries every object of $D^{>N}(\mathcal{A})$ to an object $D^{>0}(\mathcal{B})$. In particular, for every $n > N$, the complex $Ho\mathcal{F}(cone(\tau_{<n} X \rightarrow X))$ has trivial cohomology in non-positive degrees. Hence, we have

$$Hom_{D^+(\mathcal{B})}(X', Ho\mathcal{F}(cone(\tau_{<n} X \rightarrow X))) = 0.$$

\square

Combining Lemma 2.10 and Corollary 2.9 we get a fully faithful embedding

$$(2.25) \quad S : \mathcal{T}^+ \hookrightarrow D(Sh).$$

By Lemma 2.4 S carries every quasi-functor \mathcal{F} satisfying property (P') to $s(H^0\mathcal{F}) \in Sh$. This proves the second part of Theorem 2. For the first part, let $F \in Fun(\mathcal{A}, \mathcal{B})$ be a k -linear functor, and let

$$(2.26) \quad "RF" \in \mathbb{D}(D_{dg}^+(\mathcal{A})^o \otimes_k D_{dg}^+(\mathcal{B}))$$

be the “derived functor” (see (2.11)). To complete the proof of Theorem it suffices to show the following.

Lemma 2.11. *Assume that F is left exact. Then “ RF ” is an object of \mathbb{D}^{f+} and $S("RF")$ is isomorphic to $s(F)$.*

Proof. Let us show that “ RF ” satisfies property (2.14). According Remark 2.6 it will suffice to show that, for every integer n , $Y^i \in D_{dg}^{\geq n}(\mathcal{B})$ and $X \in HoC^+(\mathcal{A})$

$$H^0("RF"(X \otimes \bigoplus_i X^i)) \xrightarrow{\sim} \prod_i H^0("RF"(X \otimes X^i)).$$

We have ([Dri], §5)

$$(2.27) \quad H^0("RF"(X \otimes X')) \simeq colim_{Q_X} Hom_{D^+(\mathcal{B})}(X', F(Y)),$$

where Q_X is the filtrant category of pairs

$$(Y \in HoC_{dg}^+(\mathcal{A}), f \in Hom_{HoC_{dg}^+(\mathcal{A})}(X, Y))$$

such that $cone(f)$ is acyclic. If X is in $HoC^{\geq n}(\mathcal{A})$ the subcategory $Q'_X \subset Q_X$ formed by pairs (Y, f) with $Y \in HoC^{\geq n}(\mathcal{A})$ is cofinal in Q_X and, hence, Q_X in equation (2.27) can be replaced by Q'_X . Thus, it is enough to prove that the category Q_X has the following property: for every countable collection $w_i = (Y_i, f_i) \in Q'_X$, ($i = 1, 2, \dots$), there exists $v \in Q_X$ such that, for every i , the set $Mor_{Q_X}(w_i, v)$ is not empty. In fact, the object

$$v = (cone(\bigoplus_i X \xrightarrow{\phi} \bigoplus_i Y_i), g),$$

where $\phi_j : X \rightarrow \bigoplus_i Y_i$ equals $f_j - f_{j-1}$ and g is induced by the morphisms $X \xrightarrow{f_1} Y_1 \hookrightarrow \bigoplus_i Y_i$, does the job.

Let us show that “ RF ” satisfies property (2.15). As we explained in Remark 2.7 it suffices to show that

$$colim H^0("RF"(\tau_{<i} X \otimes X')) \xrightarrow{\sim} H^0("RF"(X \otimes X')),$$

for every $X' \in \mathcal{B}$. In fact, formula (2.27) with $Q_{\tau_{\geq i} X}$ replaced by $Q'_{\tau_{\geq i} X}$ shows that $H^0("RF"(\tau_{\geq i} X \otimes X'))$ is trivial for $i > 0$. Hence, the morphism $H^0("RF"(\tau_{<i} X \otimes X')) \rightarrow H^0("RF"(X \otimes X'))$ is an isomorphism for $i > 1$. This proves that “ RF ” belongs to \mathbb{D}^{f+} .

For the second claim, observe that the restriction $Res("RF") \in \mathbb{D}(D_{dg}^b(\mathcal{A})^o \otimes_k D_{dg}^b(\mathcal{B}))$ is the bounded “derived functor” (2.11). Thus, we are done by Lemma 2.5. \square

Proof of theorem 3. Apply Corollary 2.3 and equation (2.25).

REFERENCES

- [BV] A. Beilinson, V. Vologodsky, *A guide to Voevodsky's motives*, Geom. Funct. Anal. 17 (2008), no. 6, 1709-1787.
- [G] A. Grothendieck, *Sur quelques points d'algèbre homologique*, Tôhoku Math. J. (2) 9 (1957), 119–221.
- [Dri] V. Drinfeld, *DG quotients of DG categories*, J. Algebra 272 (2004), no. 2, 643-691.
- [KS] M. Kashiwara, P. Schapira, *Categories and sheaves*, A series of comprehensive studies in Mathematics, v. 332, Springer, (2006).
- [K1] B. Keller, *On differential graded categories*, International Congress of Mathematicians. Vol. II, 151–190, Eur. Math. Soc., Zürich, (2006).
- [K2] B. Keller, *On the cyclic homology of exact categories*, J. Pure Appl. Algebra 136 (1999), no. 1, 1–56.
- [LV] W. Lowen, M. Van den Bergh, *Hochschild cohomology of abelian categories and ringed spaces*, Adv. Math. 198 (2005), no. 1, 172–221.
- [N] A. Neeman, *The derived category of an exact category*, J. Algebra 135 (1990), no. 2, 388-394.
- [Q] D. Quillen, *Higher algebraic K-theory: I*, Higher K-Theories, Lecture Notes in Mathematics, 341 (1972), Springer, pp. 85-147.
- [T] B. Toën, *The homotopy theory of dg-categories and derived Morita theory*, Invent. Math. 167 (2007), no. 3, 615–667.
- [V] V. Vologodsky, *The Albanese functor commutes with the Hodge realization*, arXiv:0809.2830 (2009).